ON THE COMPLEXITY OF TIMETABLE AND
INTEGRAL MULTI-COMMODITY FLOW PROBLEMS

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A very primitive version of Gotlieb's timetable problem is shown to be NP-complete, and therefore all the common timetable problems are NP-complete. A polynomial time algorithm, in case all teachers are binary, is shown. The theorem that a meeting function always exists if all teachers and classes have no time constraints is proved. The multi-commodity problem is shown to be NP-complete even if the number of commodities is 3. This is true both in the directed and undirected cases. Finally, a simple reduction to the 3 dimensional matching problem is given.
I. THE TIMETABLE PROBLEM IS NP-COMPLETE

The timetable problem (TT), which we shall discuss here, is a mathematical model of the problem of scheduling the teaching program is a school. In fact it is a rather naive model since it ignores several factors which definitely play a role in practice [1]. However, we shall show that even a further restriction of the problem still leads to an NP-complete problem [2,3].

Definition of TT: Given the following data:

(1) A finite set $H$ (of hours in the week);

(2) A collection \{$T_1, T_2, \ldots, T_n$\} where $T_i \subseteq H$; (There are $n$ teachers and $T_i$ is the set of hours during which the $i$-th teacher is available for teaching.)

(3) A collection \{$C_1, C_2, \ldots, C_m$\} where $C_j \subseteq H$; (There are $m$ classes and $C_j$ is the set of hours during which the $j$-th class is available for studying.)

(4) An $n \times m$ matrix $R$ of non-negative integers; ($R_{ij}$ is the number of hours which the $i$-th teacher is required to teach the $j$-th class.)

The problem is to determine whether there exists a meeting function

$f(i,j,h) : \{1,\ldots,n\} \times \{1,\ldots,m\} \times H \rightarrow \{0,1\}$

(where $f(i,j,h) = 1$ if and only if teacher $i$ teaches class $j$ during hour $h$)

such that:
(a) \( f(i,j,h) = 1 \Rightarrow h \in T_i \cap C_j \);

(b) \( \sum_{h \in H} f(i,j,h) = R_{ij} \) for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \);

(c) \( \sum_{i=1}^{n} f(i,j,h) \leq 1 \) for all \( 1 \leq j \leq m \) and \( h \in H \);

(d) \( \sum_{j=1}^{m} f(i,j,h) \leq 1 \) for all \( 1 \leq i \leq n \) and \( h \in H \).

(a) assures that a meet takes place only when both the teacher and the class are available. (b) assures that the number of meets during the week between teacher \( i \) and class \( j \) is the required number \( R_{ij} \). (c) assures that no class has more than one teacher at a time and (d) assures that no teacher is teaching two classes simultaneously.

A teacher \( i \) is called a \textbf{k-teacher} if \( |T_i| = k \); he is called \textbf{tight} if

\[
|T_i| = \sum_{j=1}^{m} R_{ij}.
\]

**Definition of RTT:** RTT (the restricted timetable problem) is a TT problem with the following restrictions:

1. \( |H| = 3 \),
2. \( C_j = H \) for all \( 1 \leq j \leq m \),
3. Each teacher is either a tight 2-teacher or a tight 3-teacher,
4. \( R_{ij} = 0 \) or 1 for every \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \).
Clearly both the TT and the RTT problem are in the NP class. We want to show that RTT is NP-complete. In that case TT is trivially NP-complete too. We recall the 3-SAT (satisfiability of a conjunctive normal form with 3 literals per clause) is NP-complete where 3-SAT is defined as follows: Given the data

1. a set of literals \( X = \{x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\} \),
2. a family of clauses \( D_1, D_2, \ldots, D_k \) such that for every \( 1 \leq j \leq k \) \(|D_j| = 3\) and \( D_j \subseteq X \),

the problem is to determine whether there exists an assignment of values "true" and "false" to the literals, such that

(a) exactly one of \( x_i \) and \( \bar{x}_i \) is assigned "true" while the other is assigned "false",
(b) in each clause \( D_j \) there is at least one literal assigned "true".

Theorem 1: 3-SAT \( \leq \) RTT

The proof is by displaying a polynomially bounded reduction of the 3-SAT to RTT.

Let \( p_i \) be the number of times the variable \( x_i \) appears in the clauses, i.e.

\[
p_i = \sum_{j=1}^{k} |D_j \cap \{x_i, \bar{x}_i\}|
\]

For each \( x_i \), we construct a set of \( 5 \cdot p_i \) classes which will be denoted by \( C_{ab} \) where \( 1 \leq a \leq p_i \) and \( 1 \leq b \leq 5 \). In order to simplify the
exposition we shall use a graphic representation of the classes and teachers. In our graphic representation the vertices denote classhour combinations, where the rows signify the hours and the columns the classes. (See Fig.1.) The hours are \( h_1 \), \( h_2 \) and \( h_3 \). Now a

\[
\begin{array}{cccccc}
   & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\
\hline
h_1 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\hline
h_2 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\hline
h_3 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\end{array}
\]

2-teacher who is available in hours \( h_1 \) and \( h_2 \) \( (T_1 = \{h_1, h_2\}) \) and is supposed to meet once with \( C_{a_1b_1} \) and once with \( C_{a_2b_2} \) will be represented as shown in Fig. 2. The two diagonals show the only two ways possible to schedule this teacher. A 3-teacher who has to teach \( C_{a_1b_1} \), \( C_{a_2b_2} \) and

\[
\begin{array}{cccccc}
   & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\
\hline
h_1 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\hline
h_2 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\hline
h_3 & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\end{array}
\]

\( C_{p_1,5} \)

Figure 1.

Figure 2.

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$C_{a_3b_3}$ is denoted by a line with three arrows in the columns corresponding to these classes, as shown in Fig. 3. For every $1 \leq q \leq p_1$ we add two new

$\begin{align*}
\text{h}_1 & \quad \bigcirc \quad \bigcirc \quad \bigcirc \\
\text{h}_2 & \quad \bigcirc \quad \bigcirc \quad \bigcirc \\
\text{h}_3 & \quad \bigcirc \quad \bigcirc \quad \bigcirc
\end{align*}$

Figure 3.

classes, $C'_{ql}$ and $C''_{ql}$ with the structure shown in Fig. 4. There are three teachers described in the structure, two are 2-teachers and one 3-teacher. Since all these 3 teachers must teach during $h_1$, the top 3 vertices, $(h_1,C_{ql})$, $(h_1,C'_{ql})$ and $(h_1,C''_{ql})$ must be utilized.

$\begin{align*}
\text{h}_1 & \quad \bigcirc \quad \bigcirc \quad \bigcirc \\
\text{h}_2 & \quad \bigcirc \quad \bigcirc \quad \bigcirc \\
\text{h}_3 & \quad \bigcirc \quad \bigcirc \quad \bigcirc
\end{align*}$

Figure 4.
However, we have a choice of utilizing exactly one of the vertices \((h_2, C_{q1})\) and \((h_3, C_{q1})\), while leaving the other available; there are several ways to do this, as the reader may verify by himself. As far as the rest of our structure is concerned, the effect of this substructure is as follows: \((h_1, C_{q1})\) is taken and one of \((h_2, C_{q1})\) and \((h_3, C_{q1})\) is taken. Thus, we shall delete \((h_1, C_{q1})\) from our diagrams.

Consider now the structure of teachers described in Fig. 5.

![Diagram of teachers](image)

Figure 5.

Clearly, there is a 3-teacher assigned to classes \(C_{p14}, C_{11}\) and \(C_{13}\); thus, the structure described in Fig. 5 is circular. Consider now the \(p_1\) 2-teachers who are available during \(h_1\) and \(h_2\), and the \(q\)-th such teacher is assigned to classes \(C_{q3}\) and \(C_{q4}\). We claim that all these teachers must be scheduled in the same manner; that is, either all of them teach the \(C_{q3}\) classes during \(h_1\) and the \(C_{q4}\) classes during \(h_2\), or all of them...
teach the $C_{q3}$ classes during $h_2$ and $C_{q4}$ classes during $h_1$. Assume we have a schedule which does not satisfy this consistency condition. Then, there must be a $q$ such that the $q$-th teacher teaches $C_{q3}$ during $h_2$ and $C_{q4}$ during $h_1$, while the $(q+1)$-st teacher*) teaches the $C_{(q+1),3}$ during $h_1$ and $C_{(q+1),4}$ during $h_2$. In this case the 3 teacher who must teach $C_{q4}$, $C_{(q+1),1}$ and $C_{(q+1),3}$ cannot be scheduled during $h_1$ — a contradiction.

Now, for every clause $D_j = \{\xi_1, \xi_2, \xi_3\}$ we assign a 3-teacher in the following way. He is assigned to teach one class for each of the three literals. If $\xi_1 = x_i$ and this is the $q$-th appearance of this variable, then the corresponding class is $C_{q2}$, and if $\xi_1 = \bar{x}_i$ and this is the $q$-th appearance of this variable then the corresponding class is $C_{q5}$. The classes corresponding to $\xi_2$ and $\xi_3$ are defined analogously.

This completes the definition of the RTT problem. The total number of classes defined is $21 \cdot k$, and the total number of teachers is $22 \cdot k$ (15 $k$ 2-teachers and 7 $k$ 3-teachers). We claim that the given 3-SAT problem has a positive answer if and only if the RTT problem constructed above has a positive answer.

First, assume the 3-SAT problem has a positive answer. We use, now, the values of the literals in such an assignment to display a

*) Here $q+1$ should be computed conventionally, except that $p_1 + 1 = 1$, to fit the circular structure.
schedule for the constructed RTT problem — to prove that its answer is positive too.

If \( x_i \) is assigned "true" then for every \( 1 \leq q \leq p_i \) the \( q \)-th 2-teacher is scheduled to teach \( C_{q3} \) during \( h_1 \) and he teaches \( C_{q4} \) during \( h_2 \). Conversely, if \( x_i \) is assigned "false" then for every \( 1 \leq q \leq p_i \) the \( q \)-th 2-teacher is scheduled to teach \( C_{q3} \) during \( h_2 \) and \( C_{q4} \) during \( h_1 \).

In every clause \( D_j \) there is at least one literal assigned "true"; assume it is \( \xi \). If \( \xi = x_i \) and this is the \( q \)-th appearance of this variable then the 2-teacher who is supposed to teach \( C_{q2} \) and \( C_{q3} \) is scheduled to teach \( C_{q2} \) during \( h_3 \) and \( C_{q3} \) during \( h_2 \).

![Diagram](image)

Figure 6.
(In our Fig. 6 the schedule assigned to each of the 2-teachers discussed so far is shown by a heavy solid line, and the choice we avoided is shown by a dashed line. A light solid line indicates that no choice has been made yet.) The 3-teacher of \( C_{(q-1),4} \) uses \( h_1 \) to teach \( C_{(q-1),4} \) to teach \( C_{q1} \) and \( h_3 \) to teach \( C_{q3} \). (His meets are indicated by the circled vertices.) Finally, the 3-teacher corresponding to \( D_j \) uses \( h_2 \) to teach \( C_{q2} \). It remains to be shown that he can use \( h_1 \) and \( h_3 \) to teach the other two classes he is assigned to teach. Clearly, \( h_1 \) is never occupied by any other teacher in classes of types \( C_{a2} \) and \( C_{a5} \). If \( \xi' = x_r \) is another literal in \( D_j \) and it is "false", then the corresponding \( C_{a2} \) class must be taught during \( h_2 \) by the 2-teacher and \( h_3 \) remains available. Also if \( \xi' = \bar{x}_r \) and it is "false", then \( C_{a5} \) must be taught during \( h_2 \) by the 2-teacher and again \( h_3 \) remains available. Finally, if both remaining literals in \( D_j \) are "true" then for one of them we do not follow the scheme used for \( \xi \). For example, if \( \xi' = x_r \), it is "true", and this is the a-th appearance of this variable then the 2-teacher teaches \( C_{a2} \) during \( h_2 \) and \( C_{a3} \) during \( h_3 \). The 3-teacher
teaches $C_{(a-1), 4}$ during $h_1$, $C_{a1}$ during $h_3$ and $C_{a3}$ during $h_2$ (as shown in Fig. 7). Thus, $h_3$ remains available to teach $C_{a2}$, and the scheduling of the 3-teacher corresponding to $D_j$ is now easy. The other cases are similar and the reader may check them out for himself.

Second, assume the answer to the constructed RTT problem is positive, and assume we have a legal scheduling. If in the structure of $x_1$ the 2-teachers assigned to teach $C_{q3}$ and $C_{q4}$ teach $C_{q3}$ during $h_1$ and $C_{q4}$ during $h_2$ then $x_1$ is given the value "true", and if they teach $C_{q3}$ during $h_2$ and $C_{q4}$ during $h_1$ then $x_1$ is given the value "false". It remains to be shown that each clause $D_j = \{\xi_1, \xi_2, \xi_3\}$ contains at least one literal which is "true". If $\xi \in D_j$ and it is "false" then $h_2$ is used for teaching the corresponding class ($a_{a2}$ if $\xi = x_1$, and a $C_{a2}$ if $\xi = \bar{x}_1$) by the 2-teacher which teaches it and the adjacent class ($C_{a3}$ if $\xi = x$ and $C_{a4}$ if $\xi = \bar{x}_1$). Thus, if all three literal are false the 3-teacher corresponding to $D_j$ cannot have an assignment to teach its three classes, since it cannot use $h_2$.

Q.E.D.
II. THE TIMETABLE PROBLEM WITH BINARY TEACHERS IS POLYNOMICALLY SOLVABLE

Consider the TT problem with the restriction that all teachers are 2-teachers. (A 1-teacher is of no interest.) We shall show that a simple branching procedure solves the problem in polynomial time, since the branching depth is limited.

Our algorithm will determine schedules for the teachers progressively. At a given stage, when part of the teachers have been scheduled we say that a teacher is impossible if he cannot be scheduled consistently; we say that he is implied if there is only one possible way to schedule him consistently with the schedules established so far.

Algorithm:

(1) Set PHASE to 2.

(2) If all teachers have been scheduled, halt with a positive answer.

(3) If there is an unscheduled teacher who is impossible, go to (7).

(4) If there are no unscheduled implied teacher, go to (6).

(5) Let \( T_i \) be an unscheduled implied teacher. Temporarily schedule \( T_i \) as necessary and go to (2).

(6) Make all temporary schedules permanent. Let \( T_i \) be any unscheduled teacher. Arbitrarily choose a schedule for him and record this decision. Set PHASE to 1 and go to (2).

(7) If PHASE = 2, halt with a negative answer.

(8) Reverse the schedule of the recorded teacher and undo all the temporary schedules. Set PHASE to 2 and go to (3).
This algorithm clearly returns a positive answer only if a possible meeting function is constructe. It uses a limited backtracking since only one decision is ever recorded and possibly changed. Thus, the time complexity is polynomial. (In fact it is \( O(n^2) \).) It is less obvious that this limited backtracking is sufficient to discover a meeting function, if one exists.

Let a component of the evaluation be a set of teachers whose schedules gained permanency simultaneously (in Step (6)). The components may depend on arbitrary choices and on the order in which the teachers are considered. They are numbered consecutively according to their order of occurrence. For completeness, the set of teachers, who are not scheduled or their schedule had not been made permanent at the time the algorithm terminated, is considered the last component.

**Lemma 1:**

If \( T_i \) is a teacher of the last component then none of the class-hours he may use is occupied by a teacher of a previous component.

**Proof:** New component are started by entering Step (6); but this occurs only when no teacher is implied. Since all teachers are binary, the lemma follows.

Q.E.D.

The lemma implies that whenever the algorithm terminates with a negative answer, after trying both possible schedules for a certain teacher and all the schedules implied by it and failing, we can be sure
that all the permanent schedules made before could not have hindered the situation, and thus, the negative answer is conclusive.

It is worth noting here, that the technique of limited branching is applicable in other similar situations. For example one may use it to construct an $O(n)$ time complexity algorithm to solve the 2SAT (the satisfiability problem for conjunctive normal forms with at most two literals per clause). Other less efficient, but also $O(n^2)$ time complexity algorithms are the Davis and Putnam [4] algorithm, pointed out by Cook and an algorithm which follows from Quine's work [5] on the consensus (star) operation.
III. THERE IS ALWAYS A MEETING FUNCTION IF ALL TEACHERS AND CLASSES HAVE NO TIME CONSTRAINTS.

The purpose of this section is to document a theorem which follows from the classic theory of matching in bipartite graphs [6].

We say that a given TT problem has no time constraints if for all $1 \leq i \leq n$ and $1 \leq j \leq m$ $T_i = C_j = \emptyset$; we say that it is apparently feasible if the following two conditions hold:

1. For all $1 \leq i \leq n$ \[ \sum_{j=1}^{m} R_{ij} \leq |H|. \]

2. For all $1 \leq j \leq m$ \[ \sum_{i=1}^{n} R_{ij} \leq |H|. \]

Clearly the condition that a TT problem be apparently feasible is necessary for the existence of a meeting function, but is not sufficient.

Our purpose is to prove the following theorem:

Theorem 2:

If a TT problem is apparently feasible and has no time constraints then it has a meeting function.

First let us define the following quantities:

$$r = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{ij},$$

$$h = |H|,$$
\[ \nu = m - \left\lceil \frac{x}{h} \right\rceil, \]
\[ \mu = n - \left\lceil \frac{x}{h} \right\rceil. \]

Now, define a bipartite multi-graph \( G(X,Y,E) \) in the following way:

\[ X = \{x_1, x_2, \ldots, x_n\} \cup \{\xi_1, \xi_2, \ldots, \xi_\nu\}, \]
\[ Y = \{y_1, y_2, \ldots, y_m\} \cup \{\eta_1, \eta_2, \ldots, \eta_\mu\}, \]
\[ E \text{ is a set of edges connecting between vertices of } X \text{ and } \]
\[ \text{vertices of } Y \text{ constructed as follows. For every } 1 \leq i \leq n \text{ and } \]
\[ 1 \leq j \leq m \text{ we put } R_{ij} \text{ parallel edges between } x_i \text{ and } y_j. \text{ Next, for } \]
each \[ 1 \leq i \leq n \text{ we complete the degree } \ast \text{ of } x_i \text{ to be exactly } h \text{ by putting } \]
\[ h - \sum_{j=1}^{m} R_{ij} \text{ edges between } x_i \text{ and vertices of } \{\eta_1, \eta_2, \ldots, \eta_\mu\}; \]
it does not matter to which of these vertices these edges are connected provided the degree of each \( \eta_k \) never exceeds \( h \). Also, for each \[ 1 \leq j \leq m \text{ we complete the degree of } y_j \text{ to be exactly } h \text{ by putting } \]
\[ h - \sum_{i=1}^{n} R_{ij} \text{ edges between } y_j \text{ and vertices of } \{\xi_1, \xi_2, \ldots, \xi_\nu\}; \]
taking care that the degree of each \( \xi_\lambda \) never exceeds \( h \). Finally, we complete the degree of the vertices in \( \{\xi_1, \xi_2, \ldots, \xi_\nu\} \) and \( \{\eta_1, \eta_2, \ldots, \eta_\mu\} \) to be exactly \( h \) too by putting edges from any \( \xi_\lambda \) to any \( \eta_k \) which both have a lower degree.

* The degree of a vertex is the number of edges incident to it.
It remains to show that this definition is proper in the sense that all the conditions it implies are easily met.

The number of edges we construct in the completion of the degrees of \( x_1, x_2, \ldots, x_n \) is \( n \cdot h - r \). Thus, we can do this if \( \mu \cdot h \geq n \cdot h - r \), and \( \mu \) satisfies this inequality. Similarly, \( \nu \) satisfies the condition for the possibility of the completion of the degrees of \( y_1, y_2, \ldots, y_m \).

Finally, the number of edges required to complete the degrees of \( \xi_1, \xi_2, \ldots, \xi_\nu \) is \( \nu \cdot h - (m \cdot h - r) \) which is equal to \( r - \lfloor \frac{r}{h} \rfloor \cdot h \). (This is the remainder of \( r \) upon division by \( h \).) Similarly, the number of edges required to complete the degrees of \( \{n_1, n_2, \ldots, n_\mu\} \) is the same. Thus, the last part of the construction raises no difficulties either.

Next, let \( \Gamma(A) \), where \( A \subseteq X \), be the set of vertices \( B \subseteq Y \) such that there is an edge \( <a, b> \in E \) where \( a \in A \) and \( b \in B \).

**Lemma 2:**

For every \( A \subseteq X \) \[ |\Gamma(A)| \geq |A| \].

**Proof:** There are \( h \cdot |\Gamma(A)| \) edges incident to \( \Gamma(A) \) in \( G \). This includes all the edges which are incident to \( A \). Thus,

\[ h \cdot |\Gamma(A)| \geq h \cdot |A| \].

Q.E.D.

Lemma 2 assures that Hall's condition holds, and thus, by Hall's theorem [6] there is a set of \( n + \nu (= m + \mu) \) edges, no two of which have a common end point. We now use this set of edges \( M \), (which is commonly
called a complete match of $X$ to $Y$) to define the meeting function for the first hour. \( h_1 \in H; \) if \( <x_i,y_j> \in M \) then \( f(i,j,h_1) = 1; \) otherwise \( f(i,j,h_1) = 0. \) Clearly conditions (c) and (d) hold for \( h_1. \) Next we remove \( M \) from \( E. \) The new graph has degree \( h - 1 \) for all its vertices, and as in Lemma 2, Hall's condition holds again. This assures the existence of another complete match \( M' \) of \( X \) to \( Y \) and we can use it to define \( f(i,j,h_2) \) for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq m. \) We repeat this until by the \( h \)-th application all \( E \)'s edges have been used. This assures that condition (b) hold. Thus, the proof of Theorem 2 is complete.

The technique used here is an easy generalization of the one classically used to prove the school dance theorem. (See, for example, reference [7].)
IV. THE MULTI-COMMODITY FLOW PROBLEM IS NP-COMPLETE

**Definition of DMCF:** (The directed multi-commodity flow problem.)

Given the following data:

1. A finite directed graph $\mathcal{G}(V,E)$;
2. A capacity function $c(e)$ which assigns to each $e \in E$ an integer $c(e) \geq 0$;
3. A set of sources, $\{s_1, s_2, \ldots, s_k\} \subseteq V$;
4. A set of sinks, $\{t_1, t_2, \ldots, t_k\} \subseteq V$;
5. A non-negative integer $d$, called the demand.

The problem is to determine whether there exists a flow function, $\mathcal{Q}(e,i)$ which assigns to each $e \in E$ and $1 \leq i \leq k$ a non-negative integer and satisfies the following conditions:

(a) For every $e \in E$, $\sum_{i=1}^{k} \mathcal{Q}(e,i) \leq c(e)$;

(b) For every $1 \leq i \leq k$ and $v \in V - \{s_1, t_1\}$, let $e_1, e_2, \ldots, e_a$ be the edges which enter $v$ and $e'_1, e'_2, \ldots, e'_b$ the edges which emanate from $v$, then

$$\sum_{j=1}^{a} \mathcal{Q}(e_j,i) = \sum_{j=1}^{b} \mathcal{Q}(e'_j,i);$$

(This is called the conservation rule.)

(c) For every $1 \leq i \leq k$, let $e_1^i, e_2^i, \ldots, e_a^i$ be the edges which enter $s_i$ and $f_1^i, f_2^i, \ldots, f_b^i$ the edges which emanate from $s_i$, then
Knuth [8] showed that DMCF is NP-complete. Our aim is to achieve a stronger result; i.e. to prove that for 3 commodities it is already NP-complete.

**Definition of 3DMCF:** (The 3-commodity flow problem for directed graphs.)

The DMCF problem for \( k = 3 \) is called 3DMCF.

Clearly, DMCF is in the NP-class; the non-deterministic turing machine simply, "guesses" the flow function \( f(e,i) \) and proceeds to check whether it satisfies conditions (a), (b) and (c). Also, it is trivial that 3DMCF \( \leq \) DMCF. Thus, in order to show that both are NP-complete it suffices to prove the following theorem:

**Theorem 3:** \( RTT \leq 3DMCF \).

**Proof:** Given the RTT problem we construct a 3DMCF problem as follows:

\[
V = \{s_1, s_2, s_3\} \cup \{t_1, t_2, t_3\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_m\},
\]

\[
E = \{<s_h, u_i> \mid h \in T_i\} \cup \{<u_i, v_j> \mid R_{ij} = 1\}
\]

\[\cup \{<v_i, t_h> \mid 1 \leq i \leq m, \; 1 \leq h \leq 3\}.
\]

We \( \in E[c(e) = 1] \).

\[
d = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{ij}
\]
Now assume the RTT problem has a meeting function $f(i,j,h)$ which satisfies all the requirements. We assign a flow function $\mathcal{F}(e,i)$ as follows:

(i) If $h \neq k$ then $\mathcal{F}(s_h, u_i >, k) = 0$;

(ii) $\mathcal{F}(s_h, u_i >, h) = \sum_{j=1}^{m} f(i,j,h)$;

By condition (d) of the TT problem the value of this summation is either 0 or 1. It follows that condition (a) of the DMCF problem is satisfied in $<s_h, u_i >$.

(iii) $\mathcal{F}(u_i, v_j >, h) = f(i,j,h)$;

(The existence of the edge $<u_i, v_j>$ implies that $R_{ij} = 1$. Therefore, condition (b) of the TT problem assures that

$$\sum_{h=1}^{3} \mathcal{F}(u_i, v_j >, h) = 1.$$ 

Thus condition (a) of the DMCF problem is satisfied in $<u_i, v_j >$.)

(iv) If $h \neq k$ then $\mathcal{F}(v_j, t_h >, k) = 0$;

(v) $\mathcal{F}(v_j, t_h >, h) = \sum_{i=1}^{n} f(i,j,h)$;

(By condition (c) of the TT problem the value of this summation is either 0 or 1. It follows that condition (a) of the DMCF problem is satisfied in $<v_j, t_h >$.)

We have to show that the conservation rule is satisfied by $\mathcal{F}(e,i)$. If $h \neq k$ then $s_h$ obviously satisfies the rule for
commodity $k$; the same is true for the sinks $t_1, t_2$ and $t_3$.

Now, the sum of the flows of commodity $h$ into $u_i$ is

$$\varphi(<s_h, u_i>, h) = \sum_{j=1}^{m} f(i,j,h),$$

while the sum of the flows of commodity $h$ out of $u_i$ is

$$\sum_{j=1}^{m} \varphi(<u_i, v_j>, h)$$

(where $\varphi(<u_i, v_j>, h) = 0$ if $<u_i, v_j> \notin E$) which is equal to

$$\sum_{j=1}^{m} f(i,j,h).$$

Thus, the rule is satisfied at $u_i$. Finally, the sum of the flows of commodity $h$ into $v_j$ is equal to

$$\sum_{i=1}^{n} f(i,j,h),$$

while the sum of the flows of commodity $h$ out of $v_j$ is given by the same sum.

Next, we have to show that the demand is met. Consider the cut $\{<u_i, v_j> | R_{ij} = 1\}$. There are no edges in the graph from a $v_j$ to a $u_i$. All three sources are on one side of the cut, while the sinks are on the other. Thus, the total flow is equal to

$$\sum_{i=1}^{n} \sum_{j=1}^{m} 3 \varphi(<u_i, v_j>, h) = \sum_{i=1}^{n} \sum_{j=1}^{m} 3 f(i,j,h)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} R_{ij}$$

$$= d.$$
This concludes one half of the proof.

Now, assume the 3DMCF has a flow function \(\Phi(e,i)\) which satisfies the requirements of the DMCF problem. We proceed to display a meeting function \(f(i,j,h)\) for the original RTT problem.

Let

\[
f(i,j,h) = \begin{cases} 
\Phi(<u_i,v_j>,h) & \text{if } <u_i,v_j> \in E \\
0 & \text{if } <u_i,v_j> \notin E.
\end{cases}
\]

In order to show that \(f(i,j,h)\) satisfies condition (a) of the RTT problem, assume that \(f(i,j,h) = 1\) and therefore \(\Phi(<u_i,v_j>,h) = 1\). Since \(u_i\) must satisfy the conservation rule at \(u_i\), \(\Phi(<s_h,u_i>,h) = 1\). (Here we use the fact that no edges enter the sources, and therefore the only edge which can supply commodity \(h\) to \(u_i\) is \(<s_h,u_i>\).) Thus, \(<s_h,u_i> \in E\) and by \(E\)'s definition \(h \in T_i\) and \(h \in T_i \cap C_j\), as required.

The demand is met by the flow. Thus,

\[
\sum_{h=1}^{3} \Phi(<u_i,v_j>,h) = R_{ij}.
\]

This means that condition (b) of the RTT problem is satisfied.

Since \(\Phi(<v_j,t_h>,h) \leq 1\), and \(v_j,t_h\) is the only edge through which commodity \(h\) can emanate from \(v_j\), the conservation rule implies that

\[
\sum_{i=1}^{n} \Phi(<u_i,v_j>,h) \leq 1.
\]
this means that condition (c) of the TT problem is satisfied.
The proof for condition (d) is symmetric. Q.E.D.

**Definition of UMCF:** (The undirected multi-commodity flow problem.)

Given the following data:

1. A finite undirected graph $G(V,E)$;
2, 3, 4 and 5 are as in the DMCF problem.

The problem is to determine whether there exists a flow function $\varphi(<u,v>,i)$ which assigns to each $<u,v> \in E$ and $i$ an integer* and satisfies the following conditions:

(a) For every $<u,v> \in E$, $\sum_{h=1}^{k} |\varphi(<u,v>,h)| \leq c(<u,v>)$;

(b) For every $1 \leq i \leq k$ and every $v \in V - \{s_1, t_1\}$,

$$\sum_{<u,v> \in E} \varphi(<u,v>,i) = 0$$

(c) For every $1 \leq i \leq k$

$$\sum_{i=1}^{k} \sum_{<s_1,v> \in E} \varphi(<s_1,v>,i) = d.$$

* Here $\varphi(<u,v>,i) = - \varphi(<v,u>,i)$, and therefore we do not insist that the value be non-negative. Also, $c(<u,v>) = c(<v,u>)$. 
Definition of 3UMCF: (The 3-commodity flow problem for undirected graphs.)

The UMCF problem for $k = 3$ is called 3UMCF.

Again, it is clear the UMCF and 3UMCF are in NP and that $3UMCF \approx UMCF$. Thus, both are NP-complete if 3UMCF is.

We shall use a polynomial reduction of a special timetable problem (STT) to 3UMCF. The STT problem is defined as the RTT problem constructed in the proof of Theorem 1 with a slight modification. In order to assure that each class has to study in at least two of the three available hours we add several 2-teachers. The only classes in the construction of Theorem 1 which do not satisfy this condition are of types $C_{q_2}$ and $C_{q_5}$. Thus, for each $x_i$ and $1 \leq q \leq p_i$ we add a 2-teacher who is available only during $h_1$ and $h_3$ and must teach $C_{q_2}$ and $C_{q_5}$.

As the reader may verify for himself this additional teachers do not hinder the existence of a meeting function, though some local manipulation may be necessary to accommodate them. Thus, the STT problem is NP-complete too.

Now, we use a polynomial reduction from STT to 3UMCF which is completely analogous to the reduction used in the proof of Theorem 3. The part of the proof that shows that if the timetable problem has a meeting function then the demand on the flow can be met, it identical. However, in the second part, where we assume that the demand on the flow can be met and want to conclude that the timetable problem has a meeting function, an apparent difficulty is discovered. The flow may
not correspond to a legal schedule of the timetable problem since it may take advantage of the lack of direction of the edges and zigzag; i.e. there may be a flow from a \( u_i \) to an \( s_h \), or from a \( v_j \) to a \( u_i \), or from a \( t_h \) to a \( v_j \). The first of these, for example, would allow a commodity \( h \) to flow to a \( u_i \), even if there is no edge \( <s_h, u_i> \), by first flowing to some \( u_k \), back to some \( s_{k'} \), and then to \( u_i \). However, we claim that this cannot actually happen. First, there cannot be any back flow from a \( v_j \) to a \( u_i \), since the only way to meet the demand is to use all \( d \) edges of the type \( <u_i, v_j> \) from \( u_i \) to \( v_j \). Also, since the teachers are tight and the number of edges which connect \( u_i \) with the sources is exactly equal to \( |T_i| \), all of these edges must be used from the sources to \( u_i \). Finally, since each class must study at least two hours, the total flow entering \( v_j \) from the \( u_i \) type vertices is at least 2. If any of the edges from a \( t_h \) brings in flow, the 2 remaining edges to the other sinks cannot possibly satisfy the conservation rule at \( v_j \). This proves the following theorem:

**Theorem 4:** 3UMCF is NP-complete.

The 2-commodity flow problem, for undirected graphs is polynomially solvable. This can be done by searching for a minimum cut [9,10] which can be found in polynomial time [11, 12, 13]. However, in the case of directed graphs, the max-flow min-cut theorem does not hold, even for 2 commodities [14], and as far as we know, no polynomial time algorithm is known.
V. A SIMPLE REDUCTION FOR THE 3-DIMENTIONAL MATCHING PROBLEM

Karp [3] gives a proof that the 3-dimentional matching is NP-complete, through a reduction from the exact cover problem which is due to Lawler. We have found this reduction correct but hard to comprehend. Our purpose here is to point out a simple reduction from the TT problem to the 3-dimentional matching (3DM) problem.

**Definition of 3DM:** Given a set $U \subseteq D_1 \times D_2 \times D_3$ where $D_1$, $D_2$ and $D_3$ are finite sets, the problem is to determine whether there exists a matching $S \subseteq U$ such that $|S| = |D_1|$ and if $(a,b,c)$ and $(a',b',c')$ are two elements of $S$ then $a \neq a'$, $b \neq b'$ and $c \neq c'$.

**Theorem 5:** $TT \leq 3DM$.

**Proof:** Given a TT problem let us construct a 3DM as follows:

- $D_1 = \{d_{ijk} \mid \text{if } R_{ij} > 0 \text{ and } 1 \leq k \leq R_{ij}\}$;
- $D_2 = \{<i,h> \mid 1 \leq i \leq n \text{ and } h \in T_i\}$;
- $D_3 = \{<j,h> \mid 1 \leq j \leq m \text{ and } h \in C_j\}$
- $U = \{<d_{ijk},<i,h>,<j,h>> \mid d_{ijk},<i,h>,<j,h> \in D_1 \times D_2 \times D_3\}$.

The demonstration that the TT problem has a meeting function if and only if the corresponding 3DM problem has a matching is straightforward and is omitted.

Q.E.D.
For the sake of completeness we should mention here that Karp's 3DM problem is more restricted since $D_1 = D_2 = D_3$. However, our 3DM problem is reducible to his by increasing the dimensions to

$$|D_1| = |D_2| = |D_3|$$

and a proper extension of $U$.

* If $|D_1| \neq \max \{ |D_1|, |D_2|, |D_3| \}$ then clearly the answer to the 3DM problem is negative.
REFERENCES


