NUMERICAL SOLUTION FOR EIGENVALUES AND EIGENFUNCTIONS OF A HERMITIAN KERNEL AND AN ERROR ESTIMATE

by

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Technical Report No. 38
November 1974
ABSTRACT

New error estimates for eigenvalues of symmetric integral equations are obtained. These estimates are applicable to a more general class of integration methods and in many cases are better than those of Wielandt. For every eigenvalue, a numerical solution for the corresponding eigenfunction is also obtained. Whenever the exact eigenvalue happens to be simple, an error estimate for the corresponding eigenfunction is also derived.
1. Introduction

Let \( K(x,t) \) be a Hermitian kernel defined in \( I \times I \), where \( I = [a,b] \), i.e., \( K(t,x) = \overline{K(x,t)} \), such that

\[
\int_a^b |K(x,t)|^2 dt \text{ is bounded in } I;
\]

then all the characteristic values \( \mu_i \) of \( K(x,t) \) are real and there exists an orthomormal set \( \{y_i(x)\} \) of characteristic functions \( \text{c.f.} \) (see [5]), i.e.,

\[
\int_a^b K(x,t)y_i(t)dt = \mu_i y_i(x), \quad (y_i,y_j) = \delta_{ij},
\]

where \( (u,v) = \int_a^b u(x)v(x)dx \) is the scalar product of two complex functions \( u(x), v(x) \in L^2(I) = \{z(x) | (s,s) < \infty \} \).

Further, let \( S \) be a rule of numerical integration with weights \( w_{in} > 0 \) and nodes \( x_{in} \in I \), \( i = 1, \ldots, n \), by which the approximation

\[
\int_a^b f(x)dx \approx \sum_{i=1}^n w_{in}f(x_{in})
\]

is made.

To obtain a numerical solution for the c.v. of \( K(x,t) \), Wielandt [8] replaced the original problem by the sequence of eigenproblems

\[
K^{(n)} y_i^{(n)} = \mu_{in} y_i^{(n)} \quad K_{ij}^{(n)} = w_{jn} K(x_{in},x_{jn}), \quad i,j = 1, \ldots, n,
\]

with real \( \mu_{in} \) and \( n \) linearly independent eigenvectors \( y_i^{(n)} \), for a class of integration rules possessing the properties

\[
\lim_{n \to \infty} \sum_{i=1}^n w_{in}f(x_{in}) = \int_a^b f(x)dx \quad \text{for every } f(x) \in C(I),
\]

\[
\sum_{i=1}^n w_{in} = b - a;
\]

the eigenvalues \( \mu_{kn} \), \( k = 1, \ldots, n \), are then taken by Wielandt as approximations, which also converge as \( n \to \infty \), to the corresponding c.v. of \( K(x,t) \). To specify this corres-
pondence, the following assumptions are made:

Let \( V = \{ \alpha_1, \ldots, \alpha_m \} \) be a subset of the set \( R \) of all eigenvalues of a square matrix \( A \) or of all c.v. of a kernel \( F(x,t) \) defined in \( I \times I \), and let \( W = \{ z^2 | z \in V \} \); then,

(a) if \( \alpha_1, \ldots, \alpha_m \) are the \( m \) largest (smallest) real elements of \( R \) such that 
\[ \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_m (\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m), \]
then every \( \alpha_i \neq \alpha_m \) with multiplicity \( r_i > 1 \) occurs \( r_i \) times in \( V \),

(b) if \( \alpha_1, \ldots, \alpha_m \) are the \( m \) real elements of \( R \) of the largest modulus such that
\[ |\alpha_1| \geq |\alpha_2| \geq \ldots \geq |\alpha_m| \]
and there are \( r_i \) real elements of \( R \) of modulus \( |\alpha_i| \),
then every \( \alpha_i^2 \neq \alpha_m^2 \) occurs \( r_i \) times in \( W \).

The problem which arises now is what is the best error estimate for the eigenvalues \( \mu_{kn} \) of (2). In this context, and with the above assumptions, Wielandt obtained for those integration rules, which we shall call convergent with respect to \( K(x,t) \) - c.w.r.t. \( K(x,t) \) - i.e. the sequence

\[ \eta_n(x,t) = \frac{1}{n} \sum_{i=1}^{n} K(x,x_i)K(x,x_i) - \int_{I} K(x,z)K(z,t)dz \]

of the error functions for \( f(z) \neq K(x,z)K(z,t) \) converges to 0 uniformly in \( I \times I \), the following result:

Let
\[ \mu_1^+ > \mu_2^+ > \ldots \geq \mu_{kn}^+ > 0 > \mu_{sn}^- \geq \ldots \geq \mu_{2n}^- > \mu_{1n}^- \]
be the \( r \) largest positive and the \( s \) smallest negative eigenvalues of (2), and let
\[ \mu_1^+ > \mu_2^+ > \ldots \geq \mu_r^+ > 0 > \mu_1^- \geq \ldots \geq \mu_2^- \geq \mu_1^- \]
be the corresponding c.v. of \( K(x,t) \); then
\[ \mu_i^+ = \lim_{n \to \infty} \mu_{in}^+ , \quad \mu_j^- = \lim_{n \to \infty} \mu_{jn}^- , \quad i = 1, \ldots, r, \quad j = 1, \ldots, s, \]
and this convergence is uniform in i and j, i.e.

\[ \sigma_{kn} \equiv |\mu_{kn} - \mu_k| \leq q_n, \lim_{n \to \infty} q_n = 0, \]

where either

\[ \mu_{kn} = \mu_{kn}^+, \mu_k = \mu_k^+ \] or \[ \mu_{kn} = \mu_{kn}^-, \mu_k = \mu_k^- \] .

Baker [2] obtained convergence properties of a similar type for simple characteristic values of \( K(x,t) \). The best estimate obtained by Wielandt is \( q_n = O(\sqrt{\varepsilon_n}) \), where

\[ \varepsilon_n = \max_{|x|} |\eta_n(x,t)| \]

and \( \eta_n(x,t) \) is defined by (4), whereas that of Baker is \( q_n = O(\max \omega_{in}) \). Other authors ([1], [3]) obtained better bounds, but only for the distance of every eigenvalue \( \mu_{kn} \) to the nearest characteristic value of \( K(x,t) \). In this paper improved estimates of the form (see Theorem 1 at the beginning of Section 4)

\[ \sigma_{kn} = [\max(|\mu_{kn}|,|\mu_k|)]^{-1} \rho_n, \quad \rho_n = O(\varepsilon_n), \]

are obtained, which generalizes Wielandt's convergence theorems for all integration rules which are c.w.r.t. \( K(x,t) \) and satisfy (3). Moreover, the new result enables application of integration rules, which are c.w.r.t. \( K(x,t) \), to kernels which exhibit a singular behaviour in \(|x|\) and for which, therefore, no solution can be found within the scope of Wielandt's and Baker's papers (see Example 2 in Section 2). As a consequence, an error estimate for the numerical solution of (1), convergent to 0 uniformly in \( |x| \) for every integration rule which is c.w.r.t. \( K(x,t) \), is derived. The new error estimates for eigenvalues are to be interpreted as follows: our estimates are better than those of Wielandt for the first \( m_n \) eigenvalues \( \mu_{kn} \) such that \( \max(|\mu_{kn}|,|\mu_k|) > C\sqrt{\varepsilon_n} \) for some \( C > 0 \), where the sequence \( m_n \) tends to infinity; for other eigenvalues both our and Wielandt's estimates are of the same order of magnitude, namely \( O(\sqrt{\varepsilon_n}) \), and ours are not necessarily better.
2. Numerical results

To illustrate the superiority of the new error estimates given by Theorem 1 in Section 4, two numerical examples are presented; to the second of our examples Wielandt's method does not apply.

In the tables of results given below, $\mu_{ln}^+$ and $\mu_{ln}^-$ are the eigenvalues defined in Theorem 2 near end of Section 4, whereas $y_{ln}^+(x)$ and $y_{ln}^-(x)$ are the numerical solutions for characteristic functions corresponding to $\mu_{ln}^+$ and $\mu_{ln}^-$ respectively, obtained by the procedure described at the end of Section 4. The improved error estimates are those described in Section 5. The error estimates for $y_{ln}^+(x)$ and $y_{ln}^-(x)$ are those obtained by application of the remark concluding the discussion of Theorem 3 in Section 4.

Example 1: Equation

$$\int_0^1 \max(x,t)y(t)dt = \mu y(x).$$

C.v. and c.f. are:

$R^2$ and $\frac{\sqrt{2} \cosh Rx}{\cosh R}$, where $R$ is the positive root of the equation $z \tanh z = 1$; $-r_N^2$ and $\frac{\sqrt{2} \cos r_N^x}{\cos r_N}$, $N = 1,2,\ldots$, where $0 < r_1 < r_2 < \ldots$ are the positive roots of the equation $z \tan z + 1 = 0$.

The integration rule is the trapezoidal one.

To obtain $\alpha_n^+$, $\beta_n^+$ and $\gamma_n$, as defined in Section 4, put

$$A_n(z) = (n-1)z - [(n-1)z], \quad B_n(z) = 1 - A_n(z), \quad C_n(z) = A_n(z)B_n(z), \quad D_n(z) = A_n(z) - B_n(z),$$

$$h = (n-1)^{-1}, \quad F_n(z) = 1 - z - C_n(z)[3z - hD_n(z)];$$

then

$$\eta_n(x,t) = \begin{cases} \frac{h^2}{6} (3C_n(x) + F_n(x)), & x \leq t, \\ 3C_n(t) + F_n(x), & x \geq t, \end{cases}$$
which after a simple, but lengthy, calculation yields

\[ a_n^2 = \frac{h^4}{16} \sum_{k=1}^{n-1} \left( t_{kn}(t) + hC_n(t)D_n(t) + \frac{G_n(t) + 0.3tC_n(t)}{2} \right) dt, \]

where \( C_n(t) = 0.3t + P_n(t) \), and

\[ \int \eta_n^2(x,x_{in}) \, dx = \int \eta_n^2(x,x_{in}) \, dx + \int \eta_n^2(x,x_{in}) \, dx, \]

and each of the above summands is evaluated by the closed Newton-Cotes formula with 7 points. The remark at the end of Section 4 is applied with \( p=L=1 \).

For comparison with Wielandt's results the error estimates for the negative eigenvalues \( \mu_{ln}^- \) together with error estimates for the numerical solutions for \( c.f., \) are presented in the following table:

<table>
<thead>
<tr>
<th>Case</th>
<th>( \ell )</th>
<th>Error estimate for ( \mu_{ln}^- ) by Theorem 1</th>
<th>Improved error estimate for ( \mu_{ln}^- )</th>
<th>Error estimate for ( y_{ln}^-(x) )</th>
<th>Actual error for ( \mu_{ln}^- )</th>
<th>Actual maximal error for ( y_{ln}^-(x) ), ( k=0,1,\ldots,N )</th>
<th>Wielandt's estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ell = 101 )</td>
<td>1</td>
<td>7.34 \times 10^{-5}</td>
<td>7.34 \times 10^{-5}</td>
<td>0.0067</td>
<td>1.26 \times 10^{-5}</td>
<td>0.00011</td>
<td>0.00539</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6.45 \times 10^{-4}</td>
<td>3.51 \times 10^{-4}</td>
<td>0.872</td>
<td>9.22 \times 10^{-6}</td>
<td>0.000466</td>
<td>0.00539</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.00153</td>
<td>8.15 \times 10^{-4}</td>
<td>8.72 \times 10^{-6}</td>
<td>0.00104</td>
<td>0.00539</td>
<td></td>
</tr>
<tr>
<td>( \ell = 200 )</td>
<td>1</td>
<td>1.836 \times 10^{-5}</td>
<td>1.836 \times 10^{-5}</td>
<td>0.00167</td>
<td>3.14 \times 10^{-6}</td>
<td>2.77 \times 10^{-5}</td>
<td>0.00269</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.61 \times 10^{-4}</td>
<td>8.78 \times 10^{-5}</td>
<td>0.2073</td>
<td>2.306 \times 10^{-6}</td>
<td>1.17 \times 10^{-4}</td>
<td>0.00269</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3.75 \times 10^{-4}</td>
<td>2.04 \times 10^{-4}</td>
<td>2.18 \times 10^{-6}</td>
<td>1.628 \times 10^{-5}</td>
<td>2.306 \times 10^{-6}</td>
<td>1.17 \times 10^{-4}</td>
</tr>
<tr>
<td>( \ell = 1000 )</td>
<td>1</td>
<td>6.85 \times 10^{-4}</td>
<td>3.66 \times 10^{-4}</td>
<td>1.628 \times 10^{-5}</td>
<td>2.77 \times 10^{-5}</td>
<td>1.17 \times 10^{-4}</td>
<td>0.00269</td>
</tr>
</tbody>
</table>

It is to be noted that as the initial error estimates for the eigenvalues \( \mu_{ln}^- \) tend to grow with \( \ell \), they are better than those of Wielandt only for some first eigenvalues. To obtain a comparable error estimate unobtainable by Theorem 1 for other eigenvalues, the bound \( \mu_{ln}^- \) obtained in Theorem 2 with the optimal \( C = \sqrt{0.5(1+\sqrt{5})} \) can be taken.
Example 2: Equation
\[ 0^1 (1 + i \sqrt{x} - 1 \sqrt{t})^{-1} y(t) dt = u(x). \]

The exact solution is unknown.

The integration rule, which is derived by the transformation \( u = z^2 \) for the integral \( 0^1 K(x, z) K(z, t) dz \) and application of the Gauss quadrature with weights \( w_{in} \) and nodes \( \xi_{in}, i = 1, \ldots, n \), is defined by
\[ w_{in} = 2w_{in} \xi_{in}, \quad x_{in} = \xi_{in}^2, \quad i = 1, \ldots, n; \]

therefore, using Definition (4) (see [6] p. 48),
\[ \eta_n(x, t) = 2c_n \left[ \frac{2n}{\sqrt{2}n} uK(x, u^2) K(u^2, t) \right] u = \xi = \frac{2(2n)1}{\sqrt{2}n-\sqrt{2}+1} \left[ \frac{\sqrt{2}}{\sqrt{2}-\sqrt{2}+1} \right] \]
where \( c_n = \left[ (2n)^2 (2n+1)^{-1} \right] \) and \( 0 < \xi \leq \xi(x, t) < 1 \), and consequently
\[ |\eta_n(x, t)| \leq (2n)^{-2}(2n+1)^{-1} (\sqrt{1+x} + \sqrt{1+t}). \]

The error estimates, with those for \( \mu_{in}^+ (x) \) obtained by application of the remark at the end of Section 4 with \( p = 1/2 \) and \( L = 1 \), are given in the table below:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \xi )</th>
<th>Error estimate for ( \mu_{in}^+ ) by Theorem 1</th>
<th>Improved error estimate for ( \mu_{in}^+ )</th>
<th>Error estimate for ( \mu_{in}^+ (x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>1</td>
<td>2.31 \times 10^{-7}</td>
<td>2.31 \times 10^{-7}</td>
<td>8.8 \times 10^{-7}</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.015 \times 10^{-5}</td>
<td>5.08 \times 10^{-6}</td>
<td>0.00383</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.12 \times 10^{-4}</td>
<td>1.033 \times 10^{-4}</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>9.1 \times 10^{-10}</td>
<td>9.1 \times 10^{-10}</td>
<td>3.41 \times 10^{-9}</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4.10^{-8}</td>
<td>2 \times 10^{-8}</td>
<td>1.47 \times 10^{-5}</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>8.15 \times 10^{-7}</td>
<td>4.08 \times 10^{-7}</td>
<td>0.11</td>
</tr>
</tbody>
</table>

The approximations for \( n = 9 \) rounded to 10 digits are:
\[ \mu_{1n}^+ = 0.9543482459, \mu_{2n}^+ = 0.0434068611, \mu_{3n}^+ = 0.0021321407. \]
3. Numerical solution for a c.f.

Since the new results, presented in Section 2, refer also to an error estimate for the corresponding c.f., an appropriate definition of the approximate solution for a c.f., which converges to the corresponding c.f., is to be given. To obtain such a definition, observe first that by the similarity relation between the matrix \( K^{(n)} \) of (2) and the Hermitian matrix \( H \) with \( H_{ij} = K(x_{in}, x_{jn}) \sqrt{w_i w_j} \), \( i, j = 1, \ldots, n \), the eigenvector \( y_k^{(n)} \) is related to the corresponding eigenvector \( z_k \) of \( H \) by \( z_{ki} = y_{ki} \sqrt{w_i} \), \( i = 1, \ldots, n \). Further, define a new scalar product \( (u, v)^n \) of two vectors \( u, v \) in \( C_n \) — the \( n \)-dimensional complex Euclidean space — and a new norm \( |u|^n \) in \( C_n \), by

\[
(u, v)^n = \sum_{i=1}^{n} w_i u_i \overline{v_i}, \quad |u|^n = \sqrt{(u, u)^n},
\]

and denote by \( ||f|| = \sqrt{(f, f)} \) the norm of a complex function \( f(x) \); therefore, if the eigenvectors \( z_k \), \( k = 1, \ldots, n \), of \( H \) are chosen so as to form an orthonormal set, then

\[
(y_p^{(n)}, y_q^{(n)})^n = \delta_{pq}, \quad p, q = 1, \ldots, n.
\]

For every eigenvector \( y_k^{(n)} \) of (2) with \( \mu_{kn} \neq 0 \), define now the numerical solution — n.s. — \( y_{kn}(x) \) for a c.f. generated by \( y_k^{(n)} \), which also satisfies \( y_{kn}(x_{in}) = y_k^{(n)}, \)

\[
y_{kn}(x) \equiv \mu_{kn}^{-1} \sum_{j=1}^{n} w_j y_{kj}^{(n)} K(x_{in}, x_{jn}).
\]

It is natural to expect the difference between the two sides of (1), with \( \mu_k \) and \( y_k(x) \) replaced by \( \mu_{kn} \) and \( y_{kn}(x) \) respectively, to be expressible in terms of the error function (4). In fact

\[
\mu_{kn} y_{kn}(x) - \int x B K(x, t) y_{kn}(t) dt = \mu_{kn}^{-1} \sum_{j=1}^{n} w_j y_{kj}^{(n)} \eta_n(x_{in}, x_{jn}),
\]

where \( \eta_n(x, t) \) is defined by (4).
Let, further, \( \{y_m(x)\}, m = 1, \ldots r, \) form an orthonormal base of all c.f. of \( K(x, t) \) corresponding to \( \mu_k \); then for every \( n \) with \( \mu_{kn} \neq 0 \) there exist coefficients \( c_{kn}(n), m = 1, \ldots r, \) such that the error function

\[
e_{kn}(x) = y_{kn}(x) - \sum_{m=1}^{r} c_{kn}(n) y_m(x)
\]

is of minimal norm. In fact,

\[
c_{kn}(n) = (y_{kn}, y_m^*), \quad m = 1, \ldots, r,
\]

and consequently,

\[
(8) \quad \{ \begin{align*}
(e_{kn}, y) &= 0 \text{ for every c.f. } y(x) \text{ of } K(x, t) \\
&\text{corresponding to } \mu_k.
\end{align*} \}
\]

The functions \( e_{kn}(x) \) and \( \tilde{y}_{kn}(x) = y_{kn}(x) - e_{kn}(x) \) are called the error function and the c.f., respectively, associated with \( y_{kn}(x) \).

Now, if the approximate numerical solution \( y_{kn}^*(x) \) for a c.f. is taken to be of norm 1, i.e.,

\[
y_{kn}^*(x) = ||y_{kn}||^{-1} y_{kn}(x),
\]

it can be shown that the c.f. \( Y_{kn}(x) \) of norm 1 corresponding to \( \mu_k \) such that the error function \( e_{kn}(x) = y_{kn}^*(x) - Y_{kn}(x) \) is of minimal norm, assumes the form

\[
Y_{kn}(x) = \begin{cases} 
R_{kn}^{-1} \tilde{y}_{kn}(x) & R_{kn} \neq 0, \\
y_{\mu}(x) \text{ with } \mu_{\mu} = \mu_k & R_{kn} = 0,
\end{cases}
\]

where \( R_{kn} = ||\tilde{y}_{kn}|| \). Also, since by (8)

\[
||y_{kn}||^2 = R_{kn}^2 + ||e_{kn}||^2,
\]

we have
4. Error estimate and convergence

Before presenting the results, the following notation, where \( F(x,t) \) is a kernel defined in \( I \times I \), is to be introduced:

\[ \mu_k \] and \( \mu_{kn} \), \( k=1, \ldots, r \), are the \( r \) largest (smallest) c.v. of \( K(x,t) \) and the \( r \) largest (smallest) eigenvalues of (2), respectively, such that \( \mu_1 \geq \mu_{i+1} (\mu_1 \leq \mu_{i+1}) \) and \( \mu_{in} \geq \mu_{i+1,n} (\mu_{in} \leq \mu_{i+1,n}) \), \( i = 1, \ldots, r-1 \).

\( U(F) \) is the set of all c.v. of \( F(x,t) \).
\( U_n(F) \) is the set of all eigenvalues of the matrix \( F^{(n)} \) with \( F_{ij}^{(n)} = w_{jn} F(x_{in}, x_{jn}) \), \( i, j = 1, \ldots, n \).

\( \lambda_k(F) \) and \( \lambda_{kn}(F) \) are the \( k \)-th real elements of \( U(F) \) and \( U_n(F) \), respectively, in the ordering determined by that of the \( \mu_k \) and the \( \mu_{kn} \).

\( M_k(F) \) and \( M_{kn}(F) \), \( k = 1, \ldots, r \), are the moduli of the \( r \) elements of \( U(F) \) and \( U_n(F) \), respectively, of largest modulus, such that \( M_1(F) \geq M_{i+1}(F) \) and \( M_{in}(F) \geq M_{i+1,n}(F) \), \( i = 1, \ldots, r-1 \).

\[ Q(F,u) = \int_a^b \int_a^b F(x,t) u(t) \overline{u(x)} dxdt, \]
\[ Q_n(F,u) = \sum_{i,j=1}^n w_{jn} F(x_{in}, x_{jn}) u_j \overline{u_i}. \]
\[ V_k(F) = \{ Y | \int_a^b F(x,t) Y(t) dt = \lambda_k(F) Y(x), |Y| = 1 \}, \]
\[ V_{kn}(F) = \{ z | F^{(n)} z = \lambda_{kn}(F) z, |z|_n = 1 \}, \] where
\[ f_{ij} = w_{jn} F(x_{in}, x_{jn}), \quad i, j = 1, \ldots, n, \text{ and } |z|_n \text{ is defined by (5)}. \]

\[ \delta_n(F, x, t) = \sum_{i=1}^{n} w_{in} F(x, x_{in}) F(x_{in}, t) - \int_{-b}^{b} F(x, z) F(z, t) \, dz, \]

\[ D_n(F, u) = \int_{-b}^{b} \delta_n(F, x, t) u(t) u(x) \, dx \, dt, \]

\[ D_n(F, u) = \sum_{i, j=1}^{n} w_{jn} \delta(F, x_{in}, x_{jn}) u_{j} u_{i}, \]

\[ A_{kn}(F) = \left[ \max \{ \sum_{i=1}^{n} w_{in} \int_{-b}^{b} \delta_n(F, x, x_{in}) u(x) \, dx \}^2 \right]^{1/2}, \]

\[ B_{kn}(F) = \left[ \max \{ \sum_{a}^{b} \int_{-b}^{b} \delta_n(F, x, x_{in}) u(x) \, dx \}^2 \right]^{1/2}; \]

\[ \alpha_n = \left[ \int_{-b}^{b} \int_{-b}^{b} |\eta_n(x, t)|^2 \, dx \, dt \right]^{1/2}, \quad \beta_n = \left[ \sum_{i=1}^{n} w_{jn} \int_{-b}^{b} |\eta_n(x_{in}, x_{jn})|^2 \right]^{1/2}, \]

\[ \gamma_n = \left[ \sum_{i=1}^{n} w_{jn} \int_{-b}^{b} |\eta_n(x_{in}, x_{jn})|^2 \, dx \right]^{1/2}, \quad \rho_n = \max(\alpha_n, \beta_n), \text{ where } \eta_n(x, t) \text{ is defined by (4)}. \]

The new error estimates for the eigenvalues obtained in this paper are now summarized in the two following theorems, whose proofs are given in Section 6:

**Theorem 1:** If \( v_{kn} \equiv \max(|\mu_{kn}|, |\mu_{k}|) \geq C \sqrt{\rho_n} \) for some \( C > 1 \), then

\[ |\mu_{kn} - \mu_k| \leq v_{kn}^{-1} (\gamma_n + \rho_n) \left[ 1 - v_{kn}^{-2} \rho_n \right]^{-1/2} \leq v_{kn}^{-1} (\gamma_n + \rho_n) [1 - C^{-2}]^{-1/2} \]

\[ |\mu_{kn} - \mu_k| \leq \gamma_n [v_{kn}^2 - \rho_n]^{-1/2} \]

**Theorem 2:** Let \( \mu_1 > \mu_2 > \cdots > \mu_r > 0, \mu_1 \leq \mu_2 \leq \cdots \leq \mu_s < 0 \), be the \( r \) largest positive and the \( s \) smallest negative eigenvalues of (2), and let
be the corresponding c.v. of $K(x,t)$.

If the integration rule $S$ is c.w.r.t. $K(x,t)$ and satisfies (3), then

$$\lim_{n \to \infty} \mu_{i,n}^+ = \mu_i^+, \quad \lim_{n \to \infty} \mu_{j,n}^- = \mu_j^-,$$

and the convergence is uniform in $i$ and $j$, so that for every $C > 1$

$$|\mu_{i,n}^+ - \mu_i^+| \leq q_i, \quad |\mu_{j,n}^- - \mu_j^-| \leq q_j, \quad i=1,...,r, \quad j=1,...,s,$$

$$q_n = \max(C, [C^2-1]^{-\frac{1}{2}}) \sqrt{\gamma_n + \rho_n}.$$

Theorem 2 is a generalization of Wielandt's results.

The error estimates for the approximate numerical solution of (1) is given by:

**Theorem 3:** The error function $e_{kn}(x)$ satisfies (see Definition (4))

$$|e_{kn}(x)| \leq \mu_k^{-1} \left\{ \sum_{j=1}^{n} w_j n \gamma_j^{(n)} \left[ \max |\eta_n(x,x_j)\right]^{1/2} + q_{kn} I_{jn} \sqrt{F(x)} \right\} + |\nu_{kn} - \mu_k| \ G_n(x)$$

where

$$F(x) = \int_a^b |K(x,t)|^2 \ dt, \quad G_n(x) = \left[ F(x) + \eta_n(x,x) \right]^{1/2},$$

$$q_{kn} = \sup \{ |\mu_k - \lambda|^{-1} \mid \lambda \in \sigma(K), \ \lambda \neq \mu_k \},$$

$$I_{jn} = \left[ \int_a^b |\eta_n(x,x_j)|^2 dx \right]^{1/2}.$$

This bound for $e_{kn}(x)$ - and consequently that for the function $e_{kn}^*(x)$ defined by (9) - are improvements, by a factor of $O(n^{-\frac{1}{2}})$, of the error estimate for a c.f. obtained in [4].

Error estimates for the eigenvalues in special cases are given in Section 5.

An immediate consequence of Theorem 2, analogous to the one which follows from the convergence theorem in [1], is:

If the integration rule $S$ is c.w.r.t. $K(x,t)$, then

$$e_{kn}(x) \text{ and } e_{kn}^*(x) \text{ converge to } 0 \text{ uniformly in } I.$$
Remark: If $K(x,t)$ satisfies a Lipshitz condition of the form

$$\|K(u,t) - K(v,t)\| \leq L \|u-v\|^p, \quad 0 \leq p \leq 1,$$

in $I \times I$, then (see Definition (4))

$$e_{kn}(x) = [\max_{j=1}^n |y_{kn}^{(n)}(x)| + q_{kn} I_j \sqrt{F(x)}] + [\max_{m=1}^n |y_{km}^{(n)}(x)| + 2^{-p} L \max_{0 \leq m \leq n} (x_{m+1,n} - x_{mn})^p \sum_{j=1}^n w_{jn} |y_{kj}^{(n)}|]$$

where $x_{0n} = a$ and $x_{n+1,n} = b$.

Since the estimate for $e_{kn}(x)$ involves an estimate for $q_{kn}$, it can be found only if the multiplicity of $u_k$ is known. Such an estimate is obtainable, for instance, when $u_k$ is a simple c.v., and in this case we deduce by Theorem 2, (38) and Lemma 1 obtained in Section 6:

**Corollary 4:** If $u_k$ is a simple c.v. of $K(x,t)$ and the integration rule is c.w.r.t. $K(x,t)$ and satisfies (8), then for some choice of eigenvectors $y_k^{(n)}$ such that $|y_k^{(n)}|_n = 1$ (see Definition (5)),

$$y_{kn}(x) = y_k(x) \text{ uniformly in } I,$$

and so does also $\|y_{kn}\|^{-1} y_{kn}(x)$.

By definition of $y_{kn}(x)$ it follows that

$$\|y_{kn}\|^2 = \mu_{kn}^{-2} \sum_{i=1}^n w_{in} w_{jn} y_k^{(n)}(x_{in}) y_k^{(n)}(x_{jn}) G(x_{in},x_{jn})$$

where

$$G(x,t) = \int_a^b K(z,x) K(z,t) dz = \int_a^b K(x,z) K(z,t) dz.$$

In the case where $G(x,t)$ cannot be determined exactly, an approximation $c_{kn}$ of $\|y_{kn}\|$ is found by applying some quadrature formula for determining $G(x,t)$ at the points $(x_{in}, x_{jn})$; the approximate solution for a c.v. is then taken to be $c_{kn}^{-1} y_{kn}(x)$, and the error estimate is

$$\tilde{e}_{kn}(x) = |c_{kn}^{-1} y_{kn}(x) - c_{kn} y_{kn}(x)| \leq |c_{kn}^{-1} - \|y_{kn}\|^{-1}) y_{kn}(x)|$$

$$+ |e_{kn}(x)| = (c_{kn} \|y_{kn}\|)^{-1} |(c_{kn} - \|y_{kn}\|) y_{kn}(x)| + |e_{kn}^*(x)|$$
where $e_{kn}^*(x)$ is given by (9).

5. Improved error estimates for simple c.v. and for positive-definite kernels.

An error estimate for a simple c.v. can be improved if the approximate eigenvalue $u_{kn}$ satisfies the inequality

$$|u_{kn} - u_k| < \min_{i \neq k} |u_{kn} - u_i|$$

If the integration rule $S$ is c.w.r.t. $K(x,t)$ and satisfies (3), and $u_k$ is a simple c.v., then by Theorem 2 there exists an integer $N$ such that the above inequality holds for $n > N$, and by Lemma 2 (stated and proved in Section 6),

$$|u_{kn} - u_k| = \inf \{|u_{kn} - \lambda| |\lambda \in \Omega(K)| \leq |u_{kn}^{-1}| ||y_{kn}||^{-1} \gamma_n.$$  

An error estimate for a positive-definite kernel is obtained from the following theorem:

Theorem 4: Let $\tilde{u}_{kn} \in \Omega_n(K)$ and $\tilde{u}_k \in \Omega(K)$, $k=1, \ldots, n$, such that $|\tilde{u}_{kn}| = M_{kn}(K)$ and $|\tilde{u}_k| = M_k(K)$. Then (see Definition (4))

$$|\tilde{u}_{kn}^2 - \tilde{u}_k^2| \leq M_1(\eta_n) + M_1n(\eta_n), \quad k = 1, 2, \ldots,$$

where $\tilde{u}_{kn} = 0$ for $k > n$.

Corollary 2: If $K(x,t)$ is positive-definite and $u_{kn} > \min\{\lambda \in \Omega_n(K)\}$, then

$$|u_{kn}^2 - u_k^2| \leq M_1(\eta_n) + M_1n(\eta_n).$$

We also obtain

Corollary 3: $|u_{n+k}| \leq \sqrt{M_1(\eta_n)}$, $k = 1, 2, \ldots$
6. Proofs of the theorems and side results

To obtain the final results presented in Theorems 1–4 and the particular estimates for simple cv, the following lemmas will be proved, using the notation introduced at the beginning of Section 4:

**Lemma 1:** Let \( u_n = (u_{n1}, \ldots, u_{nn}) \) and \( A_n = (a_{ij}^{(n)}) \) be sequences of vectors and \( n \times n \) matrices with complex elements. Then (see Definition (5))

(a) \( |u_n|_n = \sum_{k=1}^{n} |(u_k, z_k)_n|^2 = \sum_{k=1}^{n} |(z_k, u_n)_n|^2 \) for every sequence \( z_k, k = 1, \ldots, n, \)

satisfying

\[
(z_p, z_q)_n = \delta_{pq}, \quad p, q = 1, \ldots, n.
\]

(b) If the integration rule \( \mathcal{S} \) satisfies (3), then

\[
\lim_{n \to \infty} \max_{i} |u_{ni}| = 0 \quad \text{implies} \quad \lim_{n \to \infty} \sum_{i=1}^{n} w_i u_{ni} = \lim_{n \to \infty} \sum_{i=1}^{n} w_i y_{ki}^{(n)} u_{ni} = 0, \quad k = 1, 2, \ldots
\]

and

\[
\lim_{n \to \infty} \max_{i,j} |a_{ij}^{(n)}| = 0 \quad \text{implies} \quad \lim_{n \to \infty} \sum_{i,j=1}^{n} w_i w_j a_{ij}^{(n)} = 0.
\]

**Lemma 2:** \( D_{kn} \equiv \inf \{ |u_{kn} - \lambda| : \lambda \in \mathbb{U}(K) \} \leq \frac{\gamma_n}{|u_{kn}|} \frac{1}{|y_{kn}|} \)

This lemma is a slight improvement of the result obtained in [3].

**Lemma 3:** If \( \lambda_k \equiv \lambda_k(F) = \lambda_1(F) \), where \( F(x, t) \) is a Hermitian kernel defined in \( I \times I \), then

\[
\lambda_k(\lambda_k - \lambda_{kn}(F)) \leq A_{kn}(F) [1 - \lambda_k^{-2} \max_{v_k(F)} |D_n(F, u)|]^{-\frac{1}{2}}.
\]

**Lemma 4:** If \( \lambda_{kn} \equiv \lambda_{kn}(F) = \lambda_{1n}(F) \), where \( F(x, t) \) is a Hermitian kernel defined in \( I \times I \), then

\[
\lambda_{kn}(\lambda_{kn} - \lambda_k(F)) \leq B_{kn}(F) [1 - \lambda_{kn}^{-2} \max_{v_{kn}(F)} |D_n(F, u)|]^{-\frac{1}{2}}.
\]
The following lemma is a consequence of Weyl's theorem (see [7] p. 445):

Lemma 5: Let $D(x,t) = F(x,t) - G(x,t)$, where $F(x,t)$ and $G(x,t)$ are Hermitian kernels defined in $I \times I$; then,

(a) If $Q(D,u) \geq 0$ for every $u(x)$, then

$$\lambda_k(F) \geq \lambda_k(G), \quad k = 1, 2, \ldots$$

(b) If $Q_n(D,u) \geq 0$ for every $u \in C_n$, then

$$\lambda_{kn}(F) \geq \lambda_{kn}(G), \quad k = 1, \ldots, n.$$ 

The proof of part (a) of Theorem 1 consists of the following steps:

1. Application of Lemma 3 and part (b) of Lemma 5 to obtain

$$u_k(u_k - u_{kn}) - A_{kn}(L)[1 - \nu_k^{-2} \max \{D_n(L,u)\}]^{-1} \frac{p=1}{p}$$

where

$$L(x,t) = K(x,t) - \sum (u_p - u_k) y_p(x)y_p(t).$$

2. Application of Lemma 4 and part (a) of Lemma 5 to obtain

$$u_{kn}(u_{kn} - u_k) - B_{kn}(L_n)[1 - \nu_{kn}^{-2} \max \{D_{n}(L_n,u)\}]^{-1} \frac{p=1}{p}$$

where

$$L_n(x,t) = K(x,t) - \sum (u_p - u_{kn}) y_p(x)y_p(t).$$

3. Bounding of $A_{kn}(L)$, $\max\{|D_n(L,u)|, |u \in V_k(L)|$, $B_{kn}(L_n)$ and $\max\{|D_{n}^*(L_n,u)|, |u \in V_{kn}(L_n)|$ in terms of $\nu_n$ and $\rho_n$, which is a matter of pure manipulations.

The proof of part (b) of Theorem 1 is straightforward.

Proof of Lemma 1: Put $z^l_{kl} = z^l_{ki} \sqrt{\nu_{in}}$, $v_{ni} = u_{ni} \sqrt{\nu_{in}}$; then

$$|u_n| = |v_n|, \quad |(z_k, u_n)| = |(u_n, z_k)| = \sum_{i=1}^{n} v_{ni} z^l_{kl},$$

and part (a) of the lemma is a direct consequence of (10). Part (b) follows from
\[ |\sum_{i=1}^{n} w_i u_i| \leq \sum_{i=1}^{n} w_i |u_i| \leq \max_i |u_i| \sum_{i=1}^{n} w_i, \]
\[ |\sum_{i,j=1}^{n} w_i w_j a_{ij}(n)| \leq \sum_{i,j=1}^{n} w_i w_j |a_{ij}(n)| \leq \max_{i,j} |a_{ij}(n)| \sum_{i=1}^{n} w_i, \]

and from
\[ \left( \sum_{i=1}^{n} w_i y_{k_n}^{(n)} \right)^2 \leq \sum_{i=1}^{n} w_i |u_i|^2 \leq \left( \max_i |u_i| \right)^2 \sum_{i=1}^{n} w_i, \]

which is a consequence of (6) and the Cauchy-Schwarz inequality.

**Proof of Lemma 2:** By (7) and (1)

\[ (u_{kn} - u_i)(y_{kn}, y_i) = \sum_{j=1}^{n} w_j y_{kj}^{(n)} v_{in}(x_j), \]

where, using Definition (4), \( v_{in}(t) = \int_a^b \eta_n(x,t) \overline{y_i(x)} \, dx. \)

The functions \( K(x,x_{jn}), j = 1, \ldots, n, \) and consequently all \( y_{kn}(x) \) such that \( u_{kn} \neq 0, \) satisfy the Parseval equality

\[ \sum_{i=1}^{n} (y_{kn}, y_i)^2 = \| y_{kn} \|^2. \]

Hence, by (15), the Bessel inequality, Lemma 1 and (6),

\[ D_{kn}^2 \| y_{kn} \|^2 \leq \sum_{i=1}^{n} (u_{kn} - u_i)^2 (y_{kn}, y_i)^2 \]
\[ = \sum_{i=1}^{n} w_i \sum_{j=1}^{n} w_j y_{kj}^{(n)} v_{in}(x_j) |^2 \leq \sum_{j=1}^{n} w_j y_{kn}^{(n)} \eta_n(x,x_{jn}) |^2 dx \leq \sum_{j=1}^{n} w_j y_{kn}^{(n)} \eta_n(x,x_{jn}) |^2 dx \leq u_{kn}^2, \]

which proves the lemma.

The following notations will be used in the proofs of Lemmas 3 and 4 for a Hermitian kernel \( F(x,t) \) defined in \( I \times I: \)

\[ \{ z_i(x) \} \] is an orthonormal set of all c.f. of \( F(x,t), \) i.e.

\[ \int_a^b F(x,t) z_i(t) dt = \lambda_i(F) z_i(x), \quad (z_i, z_j) = \delta_{ij}. \]
For every \( p \), \( z_p^{(n)} \) is an eigenvector of the matrix \( F^{(n)} \) with
\[
F^{(n)}_{ij} = \sum_j F(x_{in}, x_{jn}), \quad i, j = 1, \ldots, n,
\]
Corresponding to \( \lambda^{(n)}_{\text{pn}}(F) \). Those vectors may, as shown in Section 3, be bound to the relations
\[
(z_p^{(n)}, z_q^{(n)}) = \delta_{pq}, \quad p, q = 1, \ldots, n, \tag{18}
\]
which is the analogue of (6).

Further, let \( z_p^{(n)}(x), Z_p^{(n)}(x) \) and \( \varepsilon_p^{(n)}(x) \) be the n.s. for \( z_p^{(n)}(x) \) generated by \( z_p^{(n)} \) and, the c.f. and the error function associated to \( z_p^{(n)}(x) \), respectively, as defined in Section 3; then, in analogy to (8)
\[
(\varepsilon_p^{(n)}, z) = 0 \quad \text{for every } z \in \mathcal{V}_p(F). \tag{19}
\]

**Proof of Lemma 3:** Let \( Z^*_n(\cdot) \in \mathcal{V}_k(F) \) such that
\[
(Z^*_n, Z^*_n) = 0, \quad p = 1, \ldots, k-1. \tag{20}
\]

Now, in analogy to (7) (see Definition (4)),
\[
\lambda^{(n)}_{\text{pn}}(F) z_p^{(n)}(x) - \int_a^b F(x,t) z_p^{(n)}(t) dt = [\lambda^{(n)}_{\text{pn}}(F)]^{-1} \sum_{j=1}^n \sum_{j=1}^n w_{jn} z_p^{(n)}(x) \delta_{j,n} = 0, \quad p = 1, \ldots, n.
\]

Which together with (17), (19) and (20) yields (see Definition (5))
\[
\begin{align*}
\mathcal{g}_{pn} &= \sum_{i=1}^n w_{in} z_p^{(n)}(x) \varepsilon_{in}^{(n)} = \lambda^{-1}_{k} \lambda^{(n)}_{\text{pn}}(z_p^{(n)}, z^*_n) = \lambda^{-1}_{k} (\lambda^{(n)}_{\text{pn}} - \lambda^{(n)}_{k})(z_p^{(n)}, \rho_n^{(n)}), \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{g}_{pn} &= \lambda^{-1}_{k} \lambda^{(n)}_{\text{pn}}(z_p^{(n)}, z^*_n) = \lambda^{-1}_{k} \lambda^{(n)}_{\text{pn}} (Z_p^{(n)} Z^*_n + (\varepsilon_p^{(n)}, z^*_n)) = 0, \quad p = 1, \ldots, k-1.
\end{align*}
\]

Assume that \( d_n \equiv \lambda_{k}^{(n)}(\lambda^{(n)}_{k} - \lambda_{kn}) > 0 \); then
\[
|\lambda_{k}^{(n)}(\lambda^{(n)}_{k} - \lambda_{k})| > d_n, \quad p = k, \ldots, n,
\]

and by (21), (18) and Lemma 1, using Definition (5)
\[ S_n \equiv \sum_{i=1}^{n} |\mathbf{z}_{\mathbf{x}_{in}}^{k}(\mathbf{x}_{in})|^2 = n \sum_{p=k}^{n} |\mathbf{g}_{pn}|^2 \leq d_p^{-2} \sum_{p=k}^{n} |(e_p^{(n)}, \rho_n)|^2 = d_n^{-2} |\rho_n|^2. \]

Hence

\[ S_n \leq d_n^{-2} \left( A_{kn}(F) \right)^2. \]

Also, again by (17),

\[ \sum_{i=1}^{n} \mathbf{w}_{in} \mathbf{z}_{\mathbf{x}_{in}}^{p}(\mathbf{x}_{in}) \mathbf{z}_{\mathbf{x}_{in}}^{q}(\mathbf{x}_{in}) = (\lambda_p \lambda_q)^{-1} \int_a^b \int_a^b G_n(x,t) \mathbf{z}_{\mathbf{x}}^{p}(x) \mathbf{z}_{\mathbf{x}}^{q}(t) \, dx \, dt, \]

where \[ G_n(x,t) \equiv \sum_{i=1}^{n} \mathbf{w}_{in} \mathbf{F}(\mathbf{x}_{in},x) \mathbf{F}(\mathbf{x}_{in},t), \]

and

\[ G_n(x,t) = \sum_{i=1}^{n} \mathbf{w}_{in} \mathbf{F}(t,\mathbf{x}_{in}) \mathbf{F}(\mathbf{x}_{in},x) = \int_a^b \mathbf{F}(t,z) \mathbf{F}(z,x) \, dz + \delta_n(F,t,x), \]

which together with (17) implies

\[
\begin{aligned}
\sum_{i=1}^{n} \mathbf{w}_{in} \mathbf{u}(\mathbf{x}_{in}) \mathbf{v}(\mathbf{x}_{in}) &= (u,v) + \left[ \lambda_p(F) \lambda_q(F) \right]^{-1} \int_a^b \int_a^b \delta_n(F,z,t) u(t) v(z) \, dz \, dt \\
\end{aligned}
\]

for every \( u(x) \in \mathcal{V}_p(F) \) and \( v(x) \in \mathcal{V}_q(F) \) with Hermitian \( \mathbf{F}(x,t) \),

\[ S_n = 1 + \lambda_k^{-2} \mathcal{D}_n(F,\mathbf{z}^*_n), \]

which together with (22) proves the lemma.

Proof of Lemma 4: Let, using Definition (5),

\[ \zeta_p^{(n)} = \sum_{\mathbf{T}_p^{*}} (\mathbf{z}_p^{(n)}, \mathbf{z}_p^{(n)}), \quad e_p^{(n)} = \mathbf{z}_p^{(n)} - \zeta_p^{(n)}, \quad p = 1, \ldots, k-1, \]

where

\[ \mathbf{T}_p^{*} \equiv \{ p | \mu_{pn} = \mu_{ln} \}, \quad \mathbf{z}_p^{(n)} = \mathbf{z}_p(\mathbf{x}_{in}), \quad i = 1, \ldots, n. \]

Further take \( \mathbf{z}^{(n)} \in \mathcal{V}_{kn}(F) \) such that

\[ (\mathbf{z}^{(n)}, \zeta_p^{(n)}) = 0, \quad p = 1, \ldots, k-1. \]
and let \( \tilde{z}_n(x) \) be the n.s. for a c.f. of \( F(x,t) \) generated by \( \tilde{z}^{(n)} \). Since
\[
\xi_p^{(n)} = |\xi_p^{(n)}| u_p \quad \text{with} \quad u_p \in V_{\text{in}}(F) \quad \text{for each} \quad p < k,
\]
it follows by (18) that
\[
(\tilde{z}_p^{(n)}, \tilde{z}_q^{(n)})_n = 0 \quad \text{for each} \quad q \in T^*, \quad p = 1, \ldots, k-1,
\]
which together with (24) yields
\[
(\tilde{z}_p^{(n)}, \tilde{z}_p^{(n)})_n = 0, \quad p = 1, \ldots, k-1.
\]
Now by (17)
\[
g^{(n)}_{pn} \equiv (\tilde{z}_n, \tilde{z}_p) = \lambda^{-1}_p \lambda (\tilde{z}_p^{(n)}, \tilde{z}_p^{(n)})_n,
\]
where all the c.v. \( \lambda_p(F) \) with \( p > k \) are arranged in arbitrary order.

Consequently
\[
(25) \quad g^{(n)}_{pn} = 0, \quad p = 1, \ldots, k-1.
\]
Further, by (17)
\[
U^{(n)} = \sum_{\pi = 1}^{n} w_{\pi} F(x_{\pi, \alpha}, x_{\pi, \beta}) z^{(n)}_{\pi} = \lambda^{-1}_p \lambda (\tilde{z}_p^{(n)}, \tilde{z}_p^{(n)})_n,
\]
where \( \lambda_p = \lambda_p(F) \) and
\[
\rho_{\pi} = a \int_{b}^{a} \delta_{n}(F, x_{\pi, \alpha}, u) z_{p}(u) du,
\]
and
\[
r_{\pi} = \sum_{i=1}^{n} w_{i} F(x_{i, \alpha}, x_{i, \beta}) \tilde{z}^{(n)}_{i} = \sum_{i=1}^{n} w_{i} F(x_{i, \alpha}, x_{i, \beta}) \tilde{z}^{(n)}_{i} = \lambda^{-1}_k \lambda \tilde{z}^{(n)}, \quad \ell = 1, \ldots, n,
\]
which by changing the order of summation leads to (see Definition (5))
\[
(\tilde{U}^{(n)}_{p}, \tilde{z}^{(n)})_n = (\tilde{z}^{(n)}_{p}, r)_n = \lambda^{-1}_k (\tilde{z}^{(n)}_{p}, \tilde{z}^{(n)}_{p})_n,
\]
\[
(\lambda_k - \lambda_p) (\tilde{z}^{(n)}_{p}, \tilde{z}^{(n)}_{p})_n = \lambda^{-1}_p (\rho_{p}, \tilde{z}^{(n)}_{p})_n,
\]
\[
(26) \quad g^{(n)}_{pn} = \lambda^{-1}_p \lambda^{-1}_k (\tilde{z}^{(n)}_{p}, \tilde{z}^{(n)}_{p})_n = \lambda^{-1}_k (\lambda_k - \lambda_p)^{-1} (\tilde{z}^{(n)}_{p}, \rho_{p})_n.
\]
Assume that \( d_n \leq \lambda_{kn} (\lambda_{kn} - \lambda_k) > 0 \); then

\[
|\lambda_{kn} (\lambda_{kn} - \lambda_p)| \geq d_n, \quad p \geq k.
\]

Now, similar considerations lead to the Parseval equality for \( \tilde{z}_n(x) \), which is analogous to (16), and together with (25), (26) and the Bessel inequality,

\[
|\tilde{z}_n|^2 = \sum_{p=1}^n |g_{pn}|^2 \leq d_n^{-2} \sum_{p=1}^n (\tilde{z}(n), \rho_p) \leq d_n^{-2} \int_{-\infty}^{+\infty} \sum_{i=1}^n \lambda_{i,n}^{\delta_n} (F, x_{in}, u)|^2 du.
\]

Hence

\[
\|\tilde{z}_n\| \leq d_n^{-1} B_{kn}(F).
\]

Now for every two n.s. \( u_n(x) \) and \( v_n(x) \) for a c.f. of \( F(x,t) \) generated by \( u(n) \in V_{pn}(F) \) and \( v(n) \in V_{qn}(F) \) respectively, as defined in Section 3,

\[
(u_n, v_n) = \lambda_{kn}^{-1} \sum_{j=1}^n w_{jn} u_{jn}^{(n)} v_{jn}^{(n)} g_{ij}, \quad \text{where } g_{ij} = \int_{-\infty}^{+\infty} F(x, x_{in}) \overline{\overline{F}(x, x_{jn})} dx,
\]

and

\[
g_{ij} = \int_{-\infty}^{+\infty} F(x, x_{jn}) F(x, x_{in}) dx = \sum_{m=1}^n w_{im} F^{(n)}(F)^{(n)} = \delta_{jn} \lambda_{kn}^{-1} \delta_{jn} (F, x_{jn}, x_{in}),
\]

hence, changing the order of summation,

\[
(u_n, v_n) = \sum_{j=1}^n \sum_{j=1}^n w_{jn} F^{(n)}(F) u_{jn}^{(n)} v_{jn}^{(n)} g_{ij}.
\]

(28) for every two n.s. \( u_n(x) \) and \( v_n(x) \) for a c.f. of \( F(x,t) \) generated by \( u(n) \in V_{pn}(F) \) and \( v(n) \in V_{qn}(F) \) respectively, as defined in Section 3,

with Hermitian \( F(x,t) \),

\[
\|\tilde{z}_n\|^2 = \lambda_{kn}^{-2} D_{kn}(F, \tilde{z}(n)).
\]
which together with (27) proves the lemma.

Proof of lemma 5: Part (a) is a consequence of Weyl's theorem for c.v. of Hermitian kernels ([7] p. 445). To establish part (b), observe that

\[ Q_n(F,u) = v F^{*(n)} v, \quad Q_n(G,u) = v G^{*(n)} v, \]

where

\[ v_i = u_i w_i, \quad f_{ij}^{(n)} = F(x_i x_j) w_i w_j, \quad g_{ij}^{(n)} = G(x_i x_j) w_i w_j, \]

Also, \( U_n(F) \) and \( U_n(G) \) are, because of similarity between the corresponding matrices, the sets of all eigenvalues of \( f^{(n)} \) and \( g^{(n)} \) respectively. Therefore, part (b) is a consequence of Weyl's theorem for eigenvalues of Hermitian matrices, which can be derived from his original theorem by taking kernels \( F(x,t) \) of the form

\[ F(x,t) = \sum_{i=1}^{m} a_i u_i(x) u_i(t), \quad (u_i, u_j) = \delta_{ij}, \quad i, j = 1, \ldots, m. \]

The proof of Theorem 1 will be now made along the lines whose description precedes the proof of Theorem 1.

Proof of Theorem 1:

Step 1: For the kernel \( L(x,t) \) defined by (12) it follows by Lemma 3,

\[ \mu_k^{(n)} (\mu_k - \lambda_{kn} (L)) \leq A_{kn} (L) \left( 1 - \mu_k^{(n)} \frac{\max}{\nu_k (L)} |D_n (L,u)| \right)^{-1}. \]

Further, using Definition (5),

\[ \mu_k^Q_n (K,u) - \mu_k Q_n (L,u) = \sum_{p=1}^{k-1} \mu_k (u_p - u_K) |(y_p^{(n)}, u_n)|^2 \geq 0, \]

where \( y_p^{(n)} \equiv y_p (x_i) \), \( i = 1, \ldots, n \), and by part (b) of Lemma 5,

\[ \mu_k^Q_n \geq \mu_k^Q_n (L), \]

which together with (25) implies (11).

Step 2: For the kernel \( L_n (x,t) \) defined by (14) it follows by Lemma 4,
\[
\mu_k \left( \mu_n - \lambda_k (L_n) \right) \leq B_k (L_n) \left[ 1 - \mu_{kn} \max_{V_k(L_n)} |D_n^*(L_n, u)| \right]^{-\frac{1}{2}}.
\]

Now
\[
\mu_k Q(K, u) - \mu_k Q(L_n, u) = \sum_{p=1}^{k-1} \mu_k \left( \mu_{pn} - \mu_k \right) |(y_{pn}, u)|^2 \geq 0;
\]

therefore, by part (a) of Lemma 5, \( \mu_k \geq \mu_k \lambda_k (L_n) \), which together with (30) implies (13).

Step 3: Assume, with no restriction of generality, that \( \mu_{k+1} \neq \mu_k \).

Now by (23) and (1) (see Definition (4))
\[
(31) \quad s(n) \equiv \sum_{i=1}^{n} \sin \theta_L (x, x_{in}) y (x_{in}) = \delta_{pq} \left( \mu_p \mu_q \right)^{-1} \int_{a}^{b} \int_{a}^{b} \eta_n (x, t) y_p (t) y_q (s(z)) \, dz \, dt;
\]

hence, by (1)
\[
R_n (x) \equiv \sum_{i=1}^{n} \sin \theta_L (x, x_{in}) y (x_{in}) = \int_{a}^{b} \int_{a}^{b} \theta (x, t) y (t) \, dt
\]

\[
= \sum_{i=1}^{n} \sin \theta_L (x, x_{in}) y (x_{in}) - \int_{a}^{b} K(x, t) y (t) \, dt - \sum_{p=1}^{k-2} \delta_{pq} s(n) - \delta_{pq} y (x)
\]

\[
= \mu_k \left[ \varepsilon_{np} (x) - \sum_{p=1}^{k-2} \mu_{pq} \varepsilon_{np} (x) \right],
\]

where
\[
d_{pq} \equiv \mu_p - \mu_k, \quad \varepsilon_{np} (x) \equiv \int_{a}^{b} \eta_n (x, t) y_p (t) \, dt, \quad q = 1, \ldots, k-1,
\]

and
\[
(32) \begin{cases}
\rho_n (x) \equiv \int_{a}^{b} \delta_{n} (L, x, t) Y(t) \, dt = \mu_k \left[ \sum_{i=1}^{n} \sin \theta_L (x, x_{in}) Y(x_{in}) - \int_{a}^{b} L(x, t) Y(t) \, dt \right] \\
= \mu_k \sum_{q=1}^{k} \left( Y, y_q \right) R_n (x) = \mu_k \left[ \sigma_n (x) - \sum_{p=1}^{k-1} d_{np} s_{np} (x) \right],
\end{cases}
\]

where \( \sigma_n (x) \equiv \sum_{q=1}^{k} \mu_{qn} (x) \), and \( s_{np} \equiv (\sigma_n, y_p) \), \( p = 1, \ldots, k-1 \),

for every \( Y(x) \in V_k (L) \).
It may be observed that the functional

\[ ||u||_n = \left( \sum_{i=1}^{n} w_i u_i^2 \right)^{\frac{1}{2}} \]

satisfies all the requirements of a norm except that it may vanish for \( u(x) \neq 0 \); therefore

\[
\begin{cases}
||\rho_n||_n \leq ||\nu_k||_n + \sqrt{c_n}, \\
\text{where } c_n = \sum_{i=1}^{n} w_i u_i^2, \\
\sum_{i=1}^{n} u_i^2 \leq ||u||_n^2, \\
\end{cases}
\]

(33)

for every \( Y(x) \in V_k(L) \),

and by (31),

(34) \[ c_n = \sum_{p=1}^{k-1} \mu^{-2}_p s_p n_p^2 + Q(n, z), \text{ where } z(x) = \sum_{p=1}^{k-1} \mu^{-2}_p s_p n_p^2 \]

and by the Cauchy-Schwarz and Bessel inequalities it follows that for every \( Y(x) \in V_k(L) \),

(35) \[ ||\sigma_n||_n \leq ||\nu_k||_n^{-1} Y_n, \]

(36) \[ D_n(L,Y) = \mu_2^2 D_n(K,Z) \leq \mu_2^2 \rho_n \leq \rho_n, \]

for every \( Y(x) \in V_k(L) \).

By (34) and (35)

\[ c_n \leq \sum_{p=1}^{k-1} \mu^{-2}_p s_p n_p^2 + \alpha_n ||z||^2 = \sum_{p=1}^{k-1} r_p s_p n_p^2 \leq \mu^{-2}_k \alpha_n \max_r \]

where \( r_p = (1 + \mu^{-2}_p \alpha_n) \mu^{-2}_p s_p \), \( p = 1, \ldots, k-1 \). Now, if \( |\mu_k| < |\mu_{kn}| \) then \( \mu^2_k > \rho_n > \alpha_n \), therefore,
\[ r_p \leq (1 + \nu^{-2}l_k^2)(1 - \nu^{-1}l_k)^2 \leq 1, \quad p = 1, \ldots, k-1, \quad c_n \leq \nu^{-2}l_n^2 \]

which together with (32), (33) and (35) yields, since \( \delta_n(L, x, t) = \delta_n(L, t, x) \),

\[ A_{kn}(L) \leq \gamma_n + \alpha_n \leq \gamma_n + \rho_n, \]

and by (36) and (11)

\[ u_k^*(\nu_k - \nu_{kn}) \leq (\gamma_n + \rho_n)(1 - \nu_k^{-2} \rho_n)^{-1}. \]

To obtain bounds for \( B_{kn}(L) \) and for \( \max\{|D_n^*(L, u)| | u \in V_{kn}(L)\} \),

where \( L_n(x, t) \) is defined by (14), assume, with no restriction of generality, that \( u_{k+1,n} \neq u_{kn} \). Now by (28), (2) and (6) (see Definition (4))

\[ (y_p^n, y_q^n) = \delta - 1 - \nu_{kn}^{-1} n \Sigma w_j n \bar{n}(x_{in}, x_{jn}) y_p^n(n) y_q^n(n). \]

Also, by (6) it follows that every eigenvector \( y_q^n \) of (2) is also an eigenvector of the matrix \( L_n^n \) with \( L_{ij}^n = L_n(x_{in}, x_{jn}) \bar{w}_j n \) corresponding to \( \mu_{kn} \) and \( y_{qn}(x) \) is the n.s. for a c.f. of \( L_n(x, t) \) generated by it, and consequently

\[ U_q(x) = u_{kn} y_{qn}^n(x) - \int b_{kn}(x, z) y_{qn}^n(z) = \Sigma w_{j^n} y_{qi}^n l_n(x, x_{in}) - \int b_{kn}(x, z) y_{qn}^n(z) dz \]

\[ = u_{kn}^{-1} \Sigma w_{j^n} y_{qi}^n \delta(L_n, x_{jn}). \]

On the other hand, by (6), (7) and (38) (see Definition (5))

\[ U_q(x) = \mu_{qn} y_{qn}^n(x) - \int b_{qn}(x, z) y_{qn}^n(z) dz = \Sigma d_p^n [\delta_{qp} - (y_{qn}, y_{pn})] y_{qn}^n(x) \]

\[ = \mu_{qn}^{-1} [\delta_{qn}^n(x) - \Sigma \mu_{qn}^{-1} d_p^n (s_p^n, y_{pn}^n) y_{pn}^n(x)], \]

where

\[ d_p^n = \mu_{qn}^{-1} u_{kn}^{-1}, \quad s_q^n(x) = \Sigma w_{j^n} y_{qi}^n \bar{n}(x_{jn}), \quad s^n_p(x) = s_q^n(x_{jn}), \quad q = 1, \ldots, k-1, i = 1, \ldots, n. \]
Hence
\[ \rho_n^*(x) = \sum_{j=1}^n w_j n \delta(L_n, x, x_j) = w_n \left( \sum_{p=1}^{k-1} \sum_{p'=1}^n g_{pn} y_{pn}(x) \right), \]
where
\[ x \in V_{kn}(L_n), \]
and by the triangle inequality
\[ ||\rho_n^*|| \leq \sum_{p=1}^n |w_n| \sum_{n=1}^n |\epsilon_n| + \sqrt{c_n}, \]
where
\[ c_n = \sum_{p=1}^{k-1} \sum_{p'=1}^n g_{pn} y_{pn}(x)|^2 dx, \]
for every \( u \in V_{kn}(L_n) \).

Now by (38)
\[ c_n^* = \sum_{p=1}^{k-1} \sum_{p'=1}^n |g_{pn}|^2 - D_n^*(K, z), \]
where \( z = \sum_{p=1}^{k-1} \sum_{p'=1}^n g_{pn} y_{pn}(x) \),
and by the Cauchy-Schwarz inequality, (6) and Lemma 1 it follows, using Definition (5),
that for every \( u \in V_{kn}(L_n) \),
\[ Q = \sum_{p=1}^{k-1} \sum_{p'=1}^n |(u, y_{pn})|^2 \leq w_n^2 \sum_{p=1}^n |(u, y_{pn})|^2 = w_n^2, \]
\[ |\epsilon_n(x)|^2 \leq Q \sum_{j=1}^n w_j n \left| \eta_n(x, x_j) \right|^2 \leq w_n^2 \sum_{j=1}^n w_j n \left| \eta_n(x, x_j) \right|^2, \]
and by (39)
\[ D_n^*(L_n, u) = \sum_{p=1}^n |g_{pn}|^2 - D_n^*(K, v) \leq \sum_{p=1}^n g_{pn} y_{pn}(x), \]
for every \( u \in V_{kn}(L_n) \).

By (41) and (42)
\[ c_n' \leq \sum_{p=1}^{k-1} \mu_{-2}^p \frac{2}{p} |g_{pn}| \beta_n \left(1 + \mu_{-2}^p \right) \leq \mu_{kn}^{-2} \beta_n \max_{p} r'_p, \]

where \( r'_p \equiv (1 + \mu_{-2}^p \beta_n)^{-1} \mu_{-2}^p \beta_n \), \( p = 1, \ldots, k-1 \). Now, if

\[ |\mu_{kn}^2 | \geq |\mu_k|, \quad \text{then} \quad \mu_{kn}^2 \geq \rho_n \geq \beta_n, \quad \text{therefore}, \]

\[ r'_p \leq (1 + \mu_{-2}^p \beta_n)^{-2} (1 - \mu_{-2}^p \beta_n) \leq 1, \quad p = 1, \ldots, k-1, \]

\[ c_n' \leq \mu_{kn}^{-2} \beta_n, \]

which together with (39), (40) and (42) yields, since \( \delta_n(L_n, x, t) = \tilde{\delta}_n(L_n, t, x) \),

\[ B_{kn}(L_n) \leq \gamma_n + \beta_n \leq \gamma_n + \rho_n, \]

and by (43) and (13),

\[ \mu_{kn} (u_{kn} - u_k) \leq (\gamma_n + \rho_n) (1 - \mu_{kn}^{-2} \rho_n)^{-1} \]

which together with (37) proves part (a) of the Theorem. To establish part (b), observe that by the Cauchy-Schwarz inequality

\[ \max\{ |D_n(K, u)| \quad \max\{ |D_n(K, u)| \} \leq \beta_n, \quad \max\{ |D_n(K, u)| \quad \max\{ |D_n(K, u)| \} \leq \alpha_n; \]

then part (b) follows by Lemmas 3 and 4.

Proof of Theorem 2: Let, using Definition (4), \( \epsilon_{in} \equiv \max_{i} \eta(x, \in, i) \); then by Lemma 1,

\[ \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \gamma_n = 0 \quad \text{since} \quad \lim_{n \to \infty} \max_{i} |\epsilon_{in}| = 0, \quad \text{and obviously} \quad \lim_{n \to \infty} \alpha_n = 0, \]

therefore

\[ \lim_{n \to \infty} \rho_n = 0. \]
For every $k$ and $C > 1$ there exists an integer $N = N(k)$ such that
\[ |\psi_{n}^{+} | > C \sqrt{\rho_n + \gamma_n} \]
for every $n > N$, therefore by Theorem 1
\[ \lim_{n \to \infty} \mu_{kn} = \mu_{k} \]
and $\mu_{k} \mu_{kn} > 0$ for sufficiently large $n$. Now if
\[ \psi_{in}^{+} = \max (\psi_{in}^{+}, \mu_{i}^{+}) > C \sqrt{\rho_n + \gamma_n} \]
then by Theorem 1
\[ |\psi_{in}^{+} - \mu_{i}^{+} | \leq (\psi_{in}^{+})^{-1} (\rho_n + \gamma_n) (1 - C^{-2})^{-1} \]
\[ \leq C^{-1} (1 - C^{-2})^{-1} (\rho_n + \gamma_n) = \sqrt{\frac{\rho_n + \gamma_n}{C^2 - 1}} \]
and if $\psi_{in}^{+} \leq C \sqrt{\rho_n + \gamma_n}$, then $|\psi_{in}^{+} - \mu_{i}^{+} | \leq \psi_{in}^{+} \leq C \sqrt{\rho_n + \gamma_n}$, and similar results are obtained for $|\psi_{jn}^{+} - \mu_{j}^{+} |$, $j = 1, \ldots, s$, which together with (44) proves the
Theorem.

Proof of Theorem 3: From (7) it follows that $e_{kn} (x)$ satisfies the integral equation
\[ \mu_{k} e_{kn} (x) - \int_{a}^{b} k(x, t) e_{kn} (t) dt = E_{kn} (x) \]
where
\[ E_{kn} (x) \equiv \sum_{j=1}^{n} w_{kn}^{(n)} y_{kn}(n, x; x_{jn}) - (\mu_{kn} - \mu_{k}) y_{kn} (x) \]
which together with the Cauchy-Schwarz inequality yields

$$|e_{kn}(x)| \leq \left( \int_{\Omega} |E_{kn}(x)| \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla e_{kn}(x)| \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla F(x)| \right)^{\frac{1}{2}},$$

where

$$F(x) \equiv \int_{a}^{b} |k(x,t)|^2 dt.$$  

From (16) it follows that $e_{kn}(x)$ satisfies the Parseval equality, which by (8) and (15) reduces to

$$\|e_{kn}\|^2 = \sum_{\mu_k} \left( \langle e_{kn}, y_1 \rangle \right)^2 = \sum_{\mu_k} (y_{kn}, y_1)^2$$

$$= \mu_k^{-2} \sum \mu_k^{-2} \left( \prod_{i=1}^{k} - \mu_i \right)^{-2} \sum_{j=1}^{n} w_{jn} \langle k_t^{(n)} y_j^{(n)} \rangle^2$$

where

$$T_k = \{ i \mid \mu_i \neq \mu_k \}$$

Hence, by the triangle inequality

$$\|e_{kn}\| \leq \mu_k^{-1} q_{kn} \left( \int_{\Omega} |\nabla e_{kn}(x)| \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla F(x)| \right)^{\frac{1}{2}},$$

where

$$q_{kn} = \sup_{T_k} |\mu_k^{-1} \mu_i^{-1}|,$$

and by Lemma 1 and (6),

$$|y_{kn}(x)|^2 \leq \mu_k^{-2} \sum_{i=1}^{n} w_{jn} \left|k(x) \right|^2 = \mu_k^{-2} (F(x) + \eta_n(x,x)).$$
Consequently

\[ |y_{kn}(x)| \leq |y_{kn}(x) + \sum_{j=1}^{n} w_{jn} y_{kj}(n) [K(x, x_{jn}) - K(x_{sn}, x_{jn})]| = |y_{kn}(x)| \]

which together with (46) and (47) proves the theorem.

If, on the other hand, \( K(x, t) \) satisfies in \( \mathbb{R} \times \mathbb{R} \) the Lipschitz conditions

\[ |K(u, t) - K(v, t)| \leq L |u - v|^p, \]

then by the triangle inequality

\[ |y_{kn}(x)| = |y_{kn}(x) + \sum_{j=1}^{n} w_{jn} y_{kj}(n) [K(x, x_{jn}) - K(x_{sn}, x_{jn})]| \leq |y_{kn}(x)| \]

where \( x_{on} = a, x_{n+1} = b \) and \( |x_{sn} - x| = \min_{i} |x_{mn} - x| \), which together with (45), (46) and (47) proves the Remark (see Section 4).

**Proof of Theorem 4**: The degenerate kernel

\[ G_n(x, t) = \sum_{i=1}^{n} w_{in} K(x, x_{in}) K(x_{in}, t) = \sum_{i=1}^{n} w_{in} K(x_{in}, x) K(x_{in}, t), \]

is Hermitian and \( G_n(x, t) = G(x, t) + \eta_n(x, t) \), where

\[ G(x, t) = \frac{1}{b-a} \int_{a}^{b} K(x, z) K(z, t) \, dz, \]
therefore the c.v. \( v_{kn} \) of \( G_n(x,t) \), where

\[
v_{1n} > v_{2n} > \cdots > v_{nn} = v_{n+1,n} = \cdots = 0,
\]

are related to those of \( G(x,t) \), which are \( \mu_k^2 \), by the inequalities (see [7] p. 445):

\[
|v_{kn} - \mu_k^2| \leq M_1(\eta_n), \quad k = 1, 2, \ldots
\]

The \( v_{kn} \), \( k = 1, \ldots, n \), are exactly the eigenvalues of the matrix

\[
L_n \equiv \left( w_{in} G(x_{in}, x_{jn}) \right), \quad \text{which is similar to the Hermitian matrix} \quad A^{(n)}
\]

defined by

\[
A^{(n)}_{ij} \equiv \sqrt{w_i} w_j u(x_{in}, x_{jn}) = \sqrt{w_i w_j} [G_n(x_{in}, x_{jn}) - \eta_n(x_{in}, x_{jn})]
\]

Now, a procedure similar to that described in [7] for c. v. of kernels leads to the inequalities

\[
|\mu_k^2 - v_{kn}| \leq M_1(\eta_n), \quad k = 1, 2, \ldots
\]

where \( \mu_k^2 = 0 \) for \( k > n \), which together with (48) yields

\[
|\mu_k^2 - \mu_k^2| \leq M_1(\eta_n) + M_1(\eta_n), \quad k = 1, 2, \ldots
\]

Corollary 3 follows from (48).
Acknowledgments: The author is indebted to Professor I. Navot, Faculty of Electrical Engineering, Technion, I.I.T., and to Professor J. Steinberg, Department of Mathematics, Technion, I.I.T., for their care in reading the manuscript and their comments.

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