ERROR ESTIMATE FOR THE NUMERICAL SOLUTION, WITH LARGE MATRICES, OF FREDHOLM'S INTEGRAL EQUATION

by

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Technical Report No. 36
July 1974
ABSTRACT

A bound is obtained for the actual error incurred in the numerical solution of Fredholm's integral equation, using bounds for eigenvalues of a symmetric kernel.
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1. Introduction

Following [4], a numerical solution is considered for the equation

\begin{equation}
    y(x) - \lambda \int_a^b K(x,t)y(t)dt = f(x), \quad a \leq x \leq b
\end{equation}

with arbitrary complex \( f(x) \) and \( K(x,t) \) continuous in \( I = [a,b] \) and \( I \times I \), respectively. Such a solution at points \( x_{in}, \ i = 1, \ldots, n, \) of \( I \) is obtained from the system

\begin{equation}
    (I - \lambda K^{(n)}) y^{(n)} = f^{(n)},
\end{equation}

where \( K_{ij}^{(n)} = w_{jn} K(x_{in}, x_{jn}) \) and \( f_{i}^{(n)} = f(x_{in}), \ i, j = 1, \ldots, n; \)
\( w_{jn}, \ j = 1, \ldots, n, \) are the coefficients of the integration method \( S \) with nodes \( x_{in} \), the assumptions for \( K(x,t) \) (except symmetry), \( f(x), \lambda \) and \( S \) being as in [4].

In fact, let \( y^{(n)} \) be an approximate solution of (2) obtained by actual computation and

\[ \tilde{y}^{(n)}(x) \equiv f(x) + \lambda \sum_{j=1}^{n} w_{jn} \tilde{y}^{(n)}_{j} K(x, x_{jn}), \]

the approximate numerical solution; an estimate is sought for the error

\[ \tilde{e}^{(n)}(x) = \tilde{y}^{(n)}(x) - y(x). \]
The symmetric case with a special kind of quadrature formulae (namely, such that the eigenvalues $\mu^{(n)}_i$ of the matrices $K^{(n)}$ converge to the reciprocal eigenvalues $\lambda^{-1}_i$ according to Wielandt [5]) is considered in [4]. Another error estimate for the above method, which requires a bound for the maximum norm of the inverse operator in (1) or (2), is given by Anselone and Moore [1]. In this paper error estimates for large $n$, which requires manipulations with matrices smaller than that of the system (2) and no matrix inversion, are derived.

2. Error Bound

Using the notation of Anselone and Moore, put

$$e_n = \kappa_n f - K f, \quad \eta^{(n)} = \kappa_n K - K^2, \quad \rho_n = (I - \lambda K)^\gamma_n - f, \quad a_n = \|e_n\|,$$

$$A_n = \|e_n\|_2 \quad \text{where} \quad \|u\|_2 = \left(\int_a^b |u(x)|^2 dx\right)^{1/2};$$

then (see [4])

$$(I - \lambda K)^\gamma e_n = \lambda E_n, \quad \text{where} \quad E_n = \kappa_n \rho_n + e_n + \lambda \eta^{(n)} \gamma_n.$$

The error estimate given below is expressed by $\|E_n\|$ and $\|E_n\|_2$, which are, in turn, obtained in terms of $e_n$, $\eta^{(n)}$ and the $\rho_n(x_{i_n})$, $i = 1, \ldots, n$. In fact

\begin{align*}
(4) \quad \|E_n\| & \leq a_n + |\lambda| b_n + M c_n, \\
(5) \quad \|E_n\|_2 & \leq A_n + |\lambda| B_n + c_n \max_i \left(\int_a^b |K(x,x_{i_n})|^2 dx\right)^{1/2}.
\end{align*}
where

\[ b_n = \sum_{i=1}^{n} \left| w_i \gamma(i) \right| \max_{i} \left| \kappa(n)(x, x_{in}) \right|, \]

\[ B_n = \sum_{i=1}^{n} \left| w_i \gamma(i) \right| \left( \int_{a}^{b} \left| \kappa(n)(x, x_{in}) \right|^2 dx \right)^{1/2}, \]

\[ M = \max_{I \times I} K(x, t), \]

\[ c_n = \sum_{i=1}^{n} \left| w_i \rho_n(x_{in}) \right|. \]

By (3) and the Cauchy-Schwartz inequality,

\[ \| \bar{e}_n \| \leq |\lambda| \left( \| E_n \| + \| \tilde{e}_n \| \| F \| \right), \]

where \( F(x) = \left\{ \int_{a}^{b} |K(x, t)|^2 dt \right\}^{1/2}. \)

It remains now to determine a constant \( \alpha > 0 \) such that for every \( u(x) \in C(I) \)

\[ \| u \|_2 \leq \alpha \| u - \lambda K u \|_2, \]

then, by (3),

\[ \| e_n \|_2 \leq \alpha |\lambda| \| E_n \|_2, \]

and consequently, by (6),

\[ \| \tilde{e}_n \| \leq |\lambda| (\| E_n \| + |\lambda| \| F \| \| E_n \|_2), \]

with bounds for \( E_n(x) \) and \( \| E_n \|_2 \) given by (4) and (5).
For a symmetric kernel, the value of $\alpha$ in (8) was found in [4] to be $\sup_i |1 - \lambda_i^{-1}|$, where $\lambda_i$, $i = 1,2,\ldots$, are the eigenvalues. This result will now be generalized for every continuous kernel $K(x,t)$.

Putting $G = K^*K$, where $K^*(x,t) \equiv \overline{K(t,x)}$, it follows that

$$\|u - \lambda Ku\|^2_2 = \|u\|^2_2 - (Hu,u),$$

where $(f,g) = \int_a^b f(x)\overline{g(x)}dx$ is the scalar product of two complex functions $f$ and $g$, and $H = \lambda K + \bar{\lambda}K^* - |\lambda|^2G$.

Since $H(x,t)$ is Hermitian, it follows that either $(Hu,u)$ is always $\leq 0$, or there exists

$$A = \max_{\|u\|_2 = 1} (Hu,u) = (Hu_0,u_0)$$

with $\|u_0\|^2_2 = 1$. Therefore, if $\lambda$ is not an eigenvalue of $K(x,t)$, then for every $u(x)$ with $\|u\|^2_2 = 1$,

$$\|u - \lambda Ku\|^2_2 \geq 1 - (Hu_0,u_0) = 1 - A = \|u_0 - \lambda Ku_0\|^2_2 > 0,$$

and (7) holds with $\alpha = (1-A)^{-1/2}$.

3. **Bounding of $\alpha$**

To find an upper bound for $\alpha$, an upper bound $C < 1$ for $A$ is required. Now, if the function $H(x,t)$ is not sufficiently differentiable,
or the function \( G(x,t) \) defined above is unobtainable in exact form, a Hermitian sufficiently differentiable approximation \( \hat{H}(x,t) \) of \( H(x,t) \) can be found such that \( \delta(x,t) = H(x,t) - \hat{H}(x,t) \) is sufficiently small; then

\[
A = (H u_o, u_o) = (\hat{H} u_o, u_o) + (\delta u_o, u_o) \leq \lambda + \gamma,
\]

where \( \lambda = \max \| u \|_2 \) and \( \gamma = \sup \| u \|_2 = 1 \).

and similarly,

\[
\dot{\lambda} \leq A - \gamma' \quad \text{where} \quad \gamma' = \inf \| u \|_2 = 1.
\]

Since \( \delta(x,t) \) is Hermitian, it follows that \( \gamma \geq 0 \) and \( \gamma' \leq 0 \).

Now let \( \Gamma \) and \( \Gamma' \) be the upper bounds for \( \gamma \) and \( -\gamma' \) respectively, and choose \( \hat{H}(x,t) \) in such a way that

\[
\lambda \leq c < 1 - \Gamma,
\]

which by (9) and (10) can be ensured, for instance, by requiring that

\[ A + \Gamma + \Gamma' < 1. \]

To find an estimate for \( \lambda \), the following result of Wielandt [5], which holds for some classes of quadrature formulae (depending on \( \hat{H}(x,t) \)), may be used:

Let \( \mu_i^{(n)} \) be the sequences of eigenvalues of the matrices \( \hat{H}^{(n)} \), where

\[
\hat{H}^{(n)}_{ij} = \sqrt{w_{in} w_{jn}} \hat{H}(x_{in}, x_{jn}), \quad i, j = 1, \ldots, n,
\]
completed by $\hat{\mu}_i(n) = 0$ for $i > n$, and let $\hat{\lambda}_i$ be the corresponding eigenvalues of $\hat{H}(x,t)$; then $\hat{\mu}_i(n) + \hat{\lambda}_i^{-1}$ uniformly in $i$, i.e.

$$|\hat{\mu}_i(n) - \hat{\lambda}_i^{-1}| \leq q_n \quad \text{where} \quad \lim_{n \to \infty} q_n = 0.$$ 

In particular

$$\hat{\kappa} = \lim_{n \to \infty} \hat{\lambda}_n,$$

where $\hat{\lambda}_n = \max_{|z|=1} z^* \hat{H}(n) z$ is the maximal eigenvalue of $\hat{H}(n)$.

Now suppose (11) holds; then, there exists an integer $N$ such that for every $n > N$, and for a suitable choice of $c$

$$(12) \quad \hat{\lambda}_n + q_n \leq c < 1 - r$$

and consequently

$$\hat{\lambda}_n + |\hat{\lambda}_n - \hat{\lambda}| \leq \hat{\lambda}_n + q_n \leq c.$$

It thus remains to find a value of $n$ (assuming that (11) is satisfied) such that (12) holds.

4. Evaluation of $\hat{\gamma}_n(x)$ and numerical examples

To obtain an approximation to the solution of the system (2) from which
\( \gamma_n(x) \) is evaluated, the following iterative process is applied for some \( n' < n \) (see [2], p. 186),

(13) \( Y_{m+1} = Y_m - r_m - \lambda(I-\lambda K_{n'})^{-1} K_n r_m \) where \( r_m = (I-\lambda K_n)Y_m - f \)

with arbitrary \( Y_0 \), provided that

\[ |\lambda|^2 \| (I-\lambda K_{n'})^{-1} (K_n - K_{n'}) K_n \| < 1, \]

until \( \max_i |r_m(x_{in})| \) is sufficiently small for some \( m \). The first approximation \( Y_0 \) is taken in all the following numerical examples as \( (I-\lambda K_{n'})^{-1} f \).

Equation

\[ y(x) - \lambda \int_0^1 |2x - t| y(t) dt = 1 - \lambda [2x^2 + \frac{1}{2}(1-2x) |1 - 2x|] \]

Solution

\[ y(x) \equiv 1. \]

Bounds

\[ M = 2, \quad F(x) = 4x^2 - 2x + \frac{1}{3} \leq \frac{7}{3} \]

The midpoint quadrature formula is used for all cases determined by the values of \( \lambda \) in the table below. The bounds \( B(\lambda) \) for \( \| (I-\lambda K)^{-1}\| \) are those obtained by Anselone and Moore from the corresponding bounds \( B_n(\lambda) \) for \( \| (I-\lambda K_n')^{-1}\| \).
Now, recalling the definitions for \( H(x,t), \hat{H}(x,t) \) and \( \delta(x,t) \) at the end of Section 2 and at the beginning of Section 3,

\[
H(x,t) = \lambda |2x-t| + \bar{\lambda} |2t-x| - |\lambda|^2 \left( \frac{8+|x-t|^3}{6} + xt - x - t \right);
\]

\( \hat{H}(x,t) \) is obtained through approximation of the function \(|z|\) in \([-1,1]\) to a partial sum of its Fourier series, namely

\[
\hat{H}(x,t) = 2 \text{Re}(\lambda) - 8\pi^{-2}[\lambda S_N(2x-t) + \bar{\lambda} S_N(2t-x)] - |\lambda|^2 \left( \frac{8+|x-t|^3}{6} + xt - x - t \right),
\]

where

\[
S_N(z) = \sum_{k=1}^{N} \frac{(2k-1)^{-2}}{2} \cos \left( \frac{1}{2} \pi z \right),
\]

\[
\delta(x,t) = -8\pi^{-2}[\lambda R_N(2x-t) + \bar{\lambda} R_N(2t-x)], \quad \text{where} \quad R_N(z) = \sum_{k=N+1}^{\infty} \frac{(2k-1)^{-2}}{2} \cos \left( \frac{1}{2} \pi z \right).
\]

Now

\[
(\delta y, y) = -16\pi^{-2} \text{Re} \sum_{k=N+1}^{\infty} \bar{\lambda}(2k-1)^{-2}(\alpha_k' \alpha_k + \bar{\beta}_k' \bar{\beta}_k),
\]

where

\[
\alpha_k = \int_0^1 y(x) \cos \left( \frac{1}{2} \pi x \right) dx, \quad \beta_k = \int_0^1 y(x) \sin \left( \frac{1}{2} \pi x \right) dx,
\]

\[
\alpha_k' = \int_0^1 y(x) \cos \left( (2k-1) \pi x \right) dx, \quad \beta_k' = \int_0^1 y(x) \sin \left( (2k-1) \pi x \right) dx;
\]

then by the Cauchy-Schwartz and Bessel inequalities,
(δy, y) ≤ 16π⁻²|λ|(2N+1)⁻² \sum_{k=N+1}^{\infty} |\tilde{a}_k\alpha_k + \tilde{b}_k\beta_k| ≤ 16π⁻²|λ|(2N+1)⁻²[S_N^{(1)}S_N^{(2)}]^{1/2},

where

\[ S_N^{(1)} = \sum_{k=N+1}^{\infty} (|\alpha_k|^2 + |\beta_k|^2), \quad S_N^{(2)} = \sum_{k=N+1}^{\infty} (|\alpha_k'|^2 + |\beta_k'|^2), \]

and

\[ S_N^{(1)} ≤ \sum_{k=1}^{\infty} (|\alpha_{2k}|^2 + |\beta_{2k}|^2) + \sum_{k=1}^{\infty} (|\alpha_{2k-1}|^2 + |\beta_{2k-1}|^2) ≤ \|y\|_2^{2}, \]

\[ S_N^{(2)} ≤ \frac{1}{2}\|y\|_2^{2}, \]

yielding

\[ \max\{δy, y\} ≤ 8\sqrt{\pi}\,|λ|\pi^{-2}(2N+1)^{-2}. \]

Approximation of \( \max(\tilde{N}y, y)\|y\|_2 = 1 \) is effected by application of Wielandt's Theorem 7 ([5], p. 273).

Also, the error functions \( e_n(x) \) and \( \eta(x, t) \) can be explicitly given using the error formula in Section 8 of [3] with

1) \( \alpha = \beta = \omega = 0, \quad \gamma = 1, \quad \delta = 1 \) or \( 0 \),

2) the - sign in the expansion changed into + (as a result of dropping sign \( (x-t) \) from the definition of the integrand),

and

3) \( N = 4 \) with vanishing remainder term (since the integrands are piecewise polynomial functions of their arguments).
In fact,

\[ e_n(x) = -n^{-2}(f_n(x) + \frac{4}{3n}(4x-|4x-1|)B_3(\frac{1}{2} - n \min(2x,1)) + n^{-2}[\bar{B}_4(\frac{1}{2} - 2nx) - B_4(\frac{1}{2})]
+ \text{sgn}(1-4x)(\bar{B}_4(\frac{1}{2} - 2nx) - \bar{B}_4(\frac{n-1}{2}))], \]

\[ n^{(9)}(x,t) = -n^{-2}(g_n(x,t) + \frac{4}{3n} \text{sgn}(t-4x)[\bar{B}_3(\frac{1-nt}{2}) - \bar{B}_3(\frac{1}{2} - n \min(2x,1))]), \]

where \(\bar{B}_j(y)\) is the periodic extension outside \([0,1]\) of the Bernoulli polynomial \(B_j(y)\) and

\[
 f_n(x) = \begin{cases} 
 0 & \text{if } x \leq \frac{1}{2} \\
 \frac{8x-3}{24} & \text{if } x > \frac{1}{2}
\end{cases}
\]

\[
 g_n(x,t) = \begin{cases} 
 1 + 6|4x-t|[\bar{B}_2(\frac{1}{2} - 2nx) + \bar{B}_2(\frac{1-nt}{2})], & \text{if } x \leq \frac{1}{2} \\
 \frac{t}{2} + 2x-1 + 6(2-t)\bar{B}_2(\frac{1-nt}{2}), & \text{if } x > \frac{1}{2}
\end{cases}
\]

Hence,

\[ a_n < n^{-2}(s_n + \frac{5}{12}), \quad A_n < \|f_n\| + n^{-2}s_n < n^{-2}(0.27 + s_n) \]

\[ |n^{(n)}(x,x_{in})| < (48n^2)^{-1}(22-7x_{in} + \frac{32\sqrt{3}}{9n}). \]
\[
\int_0^1 |\eta(n)(x, x_{in})|^2 dx < (6n^2)^{-2} \int_0^{1/2} \left[ \left( u_{in} + \frac{4n^{-2}}{3} + 4x^2 \right) (x^2 - x + \frac{7}{48})^2 + (2x + \frac{5}{8} x_{in} - \frac{1}{4})^2 \right. \\
+ \left. C_n^2 - \frac{C_n}{n} (u_{in} + 2) \right] dx,
\]

where
\[
s_n = \frac{\sqrt{3}}{9n} + n^{-2}, \quad u_{in} = x_{in}(x_{in} - 2), \quad C_n = 1 + \frac{4\sqrt{3}}{9n}.
\]

Comparison of the new error estimates with those of [1] and with the actual error is given in the following table:

| \lambda | n  | n' | Bound for \( B_n'(\lambda) \) by [1] | Bound for \( B_n(\lambda) \) by [1] | N from (14) | m from (13) | Bound for \( A \) by [1] | Error Estimate by [1] | New Error Estimate | Actual max \( |g_n(\frac{i}{20})| \) 0≤i≤20 |
|---------|----|----|-------------------------------------|-------------------------------------|-------------|-------------|-----------------|-----------------------|-------------------|---------------------|
| 2       | 400| 90 | 40.823                              | 124.47                              | 14          | 2           | 6.8             | 0.0031                | 1.551·10⁻⁴        | 2.56·10⁻⁸          |
| 10      | 1000| 100| 50.225                              | 943.2                               | 14          | 2           | 1.25            | 0.0642                | 4.405·10⁻³        | 1.8·10⁻⁵           |
| 18      | 2000| 100| 82.45                               | uncomputable                       | 14          | 3           | 1.443           | -                     | 9.03·10⁻⁴         | 10⁻⁶               |
| 20      | 1000| 100| 95                                  |                                     | 14          | 3           | 1.51            | -                     | 0.027             | 4.3·10⁻⁶           |
| 30      | 3000| 150| 167.6                               |                                     | 14          | 4           | 1.952           | -                     | 1.322·10⁻³        | 1.265·10⁻⁸         |
| -30     | 3000| 150| 265.6                               |                                     | 14          | 4           | 2.372           | -                     | 1.675·10⁻³        | 5.3·10⁻⁸           |
| -20     | 1000| 100| 216.64                              |                                     | 14          | 4           | 3.22            | -                     | 0.006             | 3·10⁻⁷             |
References


