WHIRL DECOMPOSITION OF STOCHASTIC SYSTEMS

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Abstract

Using a technique based on "state splitting" it is shown that every n-state Markov system is decomposable into a whirl-interconnection of n-1 2-states Markov systems.

Introduction

Markov systems are dynamical devices with a finite number of internal states, transiting probabilistically from state to state when external inputs (whose variety is assumed to be finite) are fed into it. Many physical or abstract systems can be represented as Markov Systems and this is true in particular for Probabilistic Automata (see Paz (1966)).

This paper deals with a specific aspect of decomposition of Markov systems, called whirl decomposition, and based on a "state splitting" technique. It is important to be able to decompose given systems into a sub-collection of "simpler" systems (with less states than the given one) for this may facilitate the actual construction of such a system when necessary (the simpler systems can be constructed separately and in parallel and then interconnected). Also simpler systems are easier to investigate and this may facilitate the investigation of a given system.
2. The cascade Decomposition

Definition 1: A Markov system over a (finite) input alphabet $\Sigma$ is a pair $(S, \{A(\sigma)\})$ where $S$ is a finite set of states and $\{A(\sigma)\}$ is a set of matrices (representing transitions between states) such that the matrix $A(\sigma)$ is associated to the input symbol $\sigma \in \Sigma$.

Interpretation: If at time $t$ the system is in state $s_i \in S$ and receives an input $\sigma \in \Sigma$ then it moves to state $s_j$ with probability $a_{ij}(\sigma)$ where $A(\sigma) = [a_{ij}(\sigma)]$. Thus $A(\sigma)$ is an $|S|$-dimensional Markov matrix ($|S|$ denotes the number of elements in a set $S$).

Notation: If $x$ is a word in $\Sigma^*$ ($\Sigma^*$ is the set of all words or sequences of symbols over $\Sigma$ including the empty word denoted by $\lambda$) such that $x = \sigma_1 \ldots \sigma_k$ then $A(x) = A(\sigma_1) \ldots A(\sigma_k)$; $A(x) = [a_{ij}(x)]$ and $a_{ij}(x)$ is the probability of transition from state $s_i$ to state $s_j$ after the input word $x$ has been received, as one proves easily. We shall indentify in the sequel a state $s_i$ with its index $i$ for the sake of simplicity if no confusion arises.

Definition 2: A set $S'$ of states of a Markov system in a persistent sub-system if and only if the set of states which are accessible from $S'$ are in $S'$ (a state $j$ is accessible from a state $i$ if there is $x \in \Sigma^*$ such that $a_{ij}(x) > 0$).

Note that it follows from the definition 2 above that the submatrices of the matrices $A(\sigma)$ corresponding to states in $S'$ are Markov matrices.

Definition 3: Let $A = (S, \{A(\sigma)\})_{\sigma \in \Sigma}$ $A' = (S', \{A'(\sigma, i)\})_{\sigma \in \Sigma, i \in S}$ be two Markov systems. The cascade product of the two system is the system $B = (S \times S', \{B(\sigma)\})_{\sigma \in \Sigma}$ where $B(\sigma) = [b_{ik,jl}(\sigma)]$ and $b_{ik,jl}(\sigma) = a_{ij}(\sigma) a_{kl}^{-1}(\sigma, i)$.
given that \( A(\sigma) = [a_{ij}(\sigma)] \) and \( A'(\sigma,i) = [a_{kl}(\sigma,i)] \). The graphical representation of a cascade product of two stochastic systems is given in fig. 1 below.

![Diagram of a cascade product of two Markov systems](image)

**Fig. 1** Graphical representation of a cascade product of two Markov systems.

The next state of \( A \) depends on the present state of \( A \) and on the input symbol \( \sigma \in \Sigma \). The next state of \( A' \) depends on the present state of both \( A \) and \( A' \). The state of \( B \) is represented by a pair of states one from \( A \) and one from \( A' \). The cascade product can be extended to include several systems by forming the product of a product already constructed with a new system, and so on.

**Definition 4:** A Markov System \( (S, \{ C(\sigma) \}) \) is decomposable in cascade form if and only if it is isomorphic to a persistent subsystem of a cascade product of two (or more) Markov systems such that the number of states of every component in the product is smaller than \( |S| \). [Two systems are isomorphic if there exists a 1-1 mapping between their states such that the transition probabilities between corresponding states are equal].

**Definition 5:** A partition on the state set \( S \) of a system is a collection of subsets of \( S \) such that each state in \( S \) belongs to one and only one such subset. Each subset as above will be called a block of the partition. If the number of
blocks is bigger than one and smaller than the number of states then the partition is nontrivial.

It has been proved by Bacon (1964) that the following two conditions are necessary and sufficient for a system $C=(S, \{C(\sigma)\})$ to be decomposable in cascade form.

**Lumpability condition:** There exists a nontrivial partition on the state set $S$ such that for any $\sigma$, the sum of the columns of the matrix $C(\sigma)$ corresponding to any block of the partition, is a column having equal values in entries corresponding to the same block of the partition.

**Condition of separability:** There exist two nontrivial partitions on the state set $S$ such that for any $\sigma$, the sum of the columns of the matrix $C(\sigma)$ corresponding to any block of the partition, is a column having equal values in entries corresponding to the same block of the partition.

Theorem 1: Thus the following theorem holds true: A Markov system $(S, \{C(\sigma)\})$ is decomposable if and only if it satisfies the conditions of lumpability and separability with the same $\pi$ partition in both conditions.

3. **Whirl Decomposition**

A close consideration of the conditions of Theorem 1 above will show that those conditions are restrictive and they are seldom met. In order to generalize the domain of Theorem 1 Fujimoto and Fukao (1966) suggested the possibility of "state splitting" a technique well known in the deterministic automata theory. Intuitively, "state splitting" is a technique by which a state is split into several component states such that the probability of the system being in the given state
at time $t$ before the splitting, is equal to the sum of probabilities of the system being in the component states at time $t$, for all $t$. However, even if state splitting is allowed the conditions for cascade decomposability seem to be restrictive for Markov systems and it seems reasonable to assume that they cannot always be met. Note that a cascade interconnection of a sequence of systems $A_1, A_2, \ldots, A_k$ has the property that the next state of a system $A_i$ in the interconnection depends on the present input, on its present state and on the present state of all other systems $A_j$ with $j < i$, but does not depend on the present state of any system $A_j$ with $j > i$. This means that the interconnectivity in the decomposition is not maximal a fact which has some advantages from the realization point of view (i.e. in the case where such a system is to be constructed physically). We will show now that if the interconnectivity is allowed to be maximal then any $n$-state Markov system can be decomposed into a sequence of 2-state Markov systems.

**Definition 6:** Let $A = (S, \{A(\sigma, t)\}_{\sigma \in \Sigma})$ and $B = (T, \{B(\sigma, s)\}_{\sigma \in \Sigma})$ be two Markov systems.

The system $(S \times T, \{C(\sigma)\}_{\sigma \in \Sigma})$ is the **whirl interconnection** of $A$ and $B$ if $C(\sigma) = [C_{st, s', t'}(\sigma)]$ and $C_{st, s', t'}(\sigma) = a_{ss'}(\sigma, t) b_{tt'}(\sigma, s)$ (1)

Thus, in a whirl interconnection, the next state of each system depends on the present state of both systems and on the present input.

It is easily proved that whirl interconnection of two Markov-systems is a Markov-system. A whirl interconnection reduces to a cascade interconnection if
all the matrices of one of the two component systems corresponding to the same input \( \sigma \), are equal. Once the whirl interconnection of two systems is formed the resulting system can be further whirl interconnected with a third system and so on. The resulting system will be called a whirl interconnection of the sequence of systems involved. Definition 6 is illustrated in the following figure 2.

![Fig. 2 Graphical representation of a whirl interconnection of Markov systems.](image)

We are now able to state the following:

**Theorem 2:** For each \( n \)-state Markov system \( A = (S, \{A(\sigma)\}) \) there exist two systems \( B_1 \) with state set \( T_1 \) containing 2 states and \( B_2 \) with state set \( T_2 \) containing \( n-1 \) states and partitioned on the state set \( T_1 \times T_2 = T \) of their whirl interconnection \( C = (T, \{C(\sigma)\}) \) such that \( n \) states of \( C \) belonging to the same block of \( G \) are merged, then the resulting system is equivalent to the original given system \( A \).

**Proof:** Given the system \( A \) with state set \( S = \{s_1, \ldots, s_n\} \), split the state \( s_n \) into \( n-1 \) states \( s_{n-1}, \ldots, s_1 \), i.e. Let \( A' \) be a new system having state set \( S' = \{s_1, \ldots, s_{n-1}, s_n \} \) and matrices \( A'(\sigma) = \{a'_{ij}(\sigma)\} \).
Define the following two partitions over $S'$:

$$
\pi = \{s_1 \ldots s_{n-1}, s_n \ldots s_n \} \quad \tau = \{s_{n-1} s_n, s_{n-2} s_n, \ldots, s_1 s_n \}.
$$

We shall define the matrices $A'(\sigma)$ in a way such that the above two partitions will enable us to express the system $A'$ as a whirl interconnection of two systems, a two state system $B = (\pi, \{B(\sigma, \tau_j)\})$ whose state are the blocks of $\pi$, and an $n-1$-state system $B' = (\tau, B'(\sigma, \pi_1))$ whose states are the blocks of $\tau$. In addition we will require that the partition $\{s_1 \ldots s_{n-1}, s_n \ldots s_n \}$ satisfy the lumpability condition for $A'$ so that after the states $s_{n-1} s_n$ are merged, the resulting system is equivalent to the original system $A$. (It is easy to show that "satisfies the lumpability condition" is a sufficient condition for the required equivalence). In order to satisfy all the above conditions one must have that, for any fixed row in $A'(\sigma)$ say the $i$-th, the following equations hold:

(a) \[ \sum_{s_j \in \pi_k} a'_{ij}(\sigma) = \sum_{s_j \in \tau_k} a'_{ij}(\sigma) = a_{it}(\sigma) \] with $s_t = \pi_k \cap \tau_k$

(b) \[ a'_{ij}(\sigma) = \begin{cases} a_{in}(\sigma) & \text{if } i \leq n \\ a_{nn}(\sigma) & \text{if } i > n \end{cases} \]

(c) \[ a'_{ij}(\sigma) = \begin{cases} a_{ij}(\sigma) & \text{if } i, j \leq n-1 \\ a_{nj}(\sigma) & \text{if } i > n, j \leq n-1 \end{cases} \]

Equations (b) and (c) are necessary and sufficient for the lumpability requirement while equation (a) is equivalent to the property (1). This follows from the fact that $\sum_{s_t' \in \pi'} c_{st', s_t'}(\sigma) = a_{st}(\sigma, t)$ in (1) is equivalent to $\sum_{s_j \in \tau_k} a'_{ij}(\sigma)$ here. Combining these two equations one gets from (1) that:

$$
\sum_{s_t' \in \pi'} c_{st', s_t'}(\sigma) = \sum_{s_j \in \tau_k} a'_{ij}(\sigma)
$$

which is equivalent to the equation...
(a) here. Now equations (a), (b) and (c) above uniquely determine the matrix
A'(σ) given the matrix A(σ). Indeed for i≤n, k≥n-1 we have by (a) that:
\[
\sum_{j \in \Pi_2} a'_{ij}(σ) (a'_{ik}(σ) + a'_{i,k+n-1}(σ)) = a'_{i,k+n-1}(σ)
\]
(2)

Using (b) and (c) we change this equation into the following equation where
a'_{i,k+n-1}(σ) is unknown and all the other values are known:
\[
a_{in}(σ) (a'_{ik}(σ) + a'_{i,k+n-1}(σ)) = a'_{i,k+n-1}(σ)
\]
(3)
or, by transposing the second left summand to the right hand side we have:
\[
a_{in}(σ) a'_{ik}(σ) = a'_{i,k+n-1}(σ) [1-a_{in}(σ)]
\]
(4)
thus
\[
a'_{i,k+n-1}(σ) = a_{in}(σ) \frac{a'_{ik}(σ)}{1-a_{in}(σ)}
\]
(5)
and, as 1 - a_{in}(σ) = \sum_{j<n} a_{ij}(σ) ≥ a_{ik}(σ),
both sides of the equation are non-negative. If a_{in}(σ) = 0 then a'_{i,k+n-1}(σ) = 0
and if a_{in}(σ) = 1 then a'_{i,k+n-1}(σ) can be arbitrarily chosen provided that
\[
\sum_{k=1}^{n-1} a'_{i,k+n-1}(σ) = 1
\]
and all the summand are non-negative. It follows that
if the values a'_{i,k+n-1}(σ), i≤n, k≥n-1 are chosen according to (5) then the
requirements (a), (b) and (c) are satisfied, for the derivation of (5) is
 reversible and (2) implies also the following:
\[
(\sum_{j \in \Pi_1} a'_{ij}(σ)) (a'_{ik}(σ) + a'_{i,k+n-1}(σ)) =
\]
\[
(1-\sum_{j \in \Pi_2} a'_{ij}(σ)) (a'_{ik}(σ) + a'_{i,k+n-1}(σ)) =
\]
\[
a'_{ik}(σ) + a'_{i,k+n-1}(σ) - (\sum_{j \in \Pi_2} a'_{ij}(σ)) (a'_{ik}(σ) + a'_{i,k+n-1}(σ)) =
\]
\[
a'_{ik}(σ) + a'_{i,k+n-1}(σ) - a'_{i,k+n-1}(σ) = a'_{ik}(σ)
\]
as required. As for the case $i > n$ it follows from (c) that the first $n-1$
entries in each such row must be equal to the corresponding entry in the $n$-th
row and therefore by (5) this must be true for the full rows, i.e. the $n$-th row
in $A'(\sigma)$ as determined by (c) and (5) must be duplicated $n-1$ times. It thus
follows from the construction that the system $A'(S', \{A'(\sigma)\})$ can be represented
as a whirl interconnection of the two systems $B = (T_1, \{B(\sigma, \tau_1)\})$ and $B' = (T_2, \{B(\sigma, \tau_2)\})$
where the elements of $T_1$ and $T_2$ are the blocks of $\pi$
and $\tau$ respectively and the matrices $B(\sigma, \tau_i) = [b_{k\ell}(\sigma, \tau_i)]$ and
$B'(\sigma, \pi_j) = [b'_{k\ell}(\sigma, \pi_j)]$ are defined by $b_{k\ell}(\sigma, \tau_i) = \sum_{p \in \pi_k} a'_{mp}(\sigma)$, $s_k = \tau_i \cap \pi_k$,
and
\[ b'_{k\ell}(\sigma, \pi_j) = \sum_{p \in \pi_j} a'_{mp}(\sigma) \quad s_k = \pi_j \cap \tau_k \]
\[ k, \ell = 1, 2, \ldots, n-1 \]
\[ j = 1, 2 \ldots, n \]
One sees easily from the construction that if $\rho$ is the partition $\rho = \{s_1\}$,
$\ldots, \{s_{n-1}\}, \{s_n\}$ then the system $A'$ is equivalent to $A$ when all states in a
block of $A'$ are merged into a single state q.e.d.

**Corollary 3:** For each $n$-state Markov system $A = (S, \{A(\sigma)\})$ there exist
$n-1$, 2-state systems $B_i$ with state sets $T_i$ respectively and a partition $\rho$
on the state set $T = T_1 \times T_2 \times \ldots \times T_{n-1}$ of their whirl interconnection $C = (T, \{C(\sigma)\})$
such that if states of $C$ belonging to the same block of $\rho$ are merged, then the
resulting system is equivalent to the original given system $A$. 
Proof: By Theorem 2 and induction

4. Example

Let \( A = (S, \{A(a)\}) \) be the 3-state system over \( \Sigma = \{a, b\} \) with

\[
A(a) = \begin{bmatrix}
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
0 & 1 & 0
\end{bmatrix}
\]

\[
A(b) = \begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0 \\
0 & \frac{1}{4} & \frac{3}{4}
\end{bmatrix}
\]

Using (b), (c) and (5) we construct the system \( A' = (S', \{A'(a)\}) \) with

\[
A'(a) = \begin{bmatrix}
\frac{1}{2} & \frac{1}{4} & \frac{1}{6} & \frac{1}{12} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

\[
A'(b) = \begin{bmatrix}
0 & 0 & 1 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{4} & 0 & \frac{3}{4} \\
0 & \frac{1}{4} & 0 & \frac{3}{4}
\end{bmatrix}
\]

so that \( \rho = \{(s'_1), \{s'_2\}\{s_3^1, s_3^2\}\) and \( A' \) is equivalent to \( A \) if states \( s_3^1 \) and \( s_3^2 \) are merged. Let \( \pi = (\pi_1, \pi_2) = ((s_1, s_2^1), (s_3^1, s_3^2)) \) and \( \tau = (\tau_1, \tau_2) = (s_1, s_3^1), (s_2, s_3^2) \). Using these partitions and the method outlined in the proof of Theorem 2, the systems \( B = (T_1\{B(\pi, \tau)\}) \) and \( B' = (T_2\{B'(\sigma, \tau')\}) \) are derived where

\[
B(a, \tau_1) = \begin{bmatrix}
\frac{3}{4} & \frac{1}{4} \\
1 & 0
\end{bmatrix}
\]

\[
B(a, \tau_2) = \begin{bmatrix}
\frac{2}{3} & \frac{1}{3} \\
1 & 0
\end{bmatrix}
\]

\[
B(b, \tau_1) = \begin{bmatrix}
0 & 1 \\
\frac{1}{4} & \frac{3}{4}
\end{bmatrix}
\]

\[
B(b, \tau_2) = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{4} & \frac{3}{4}
\end{bmatrix}
\]
and

\[
B'(a, \pi_1) = \begin{bmatrix}
\frac{2}{3} & 1 \\
1 & 2 & 1
\end{bmatrix}
\]

\[
B'(a, \pi_2) = \begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix}
\]

\[
B'(b, \pi_1) = \begin{bmatrix}
1 & 0 \\
1 & 0
\end{bmatrix}
\]

\[
B'(b, \pi_2) = \begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix}
\]

BIBLIOGRAPHY

