ERROR ESTIMATE FOR THE NUMERICAL SOLUTION OF FREDHOLM'S INTEGRAL EQUATION

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E. Rakotch

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ABSTRACT

A bound on the actual error incurred in the numerical solution of Fredholm's integral equation, using bounds for eigenvalues of a symmetric kernel, is obtained.

1. Introduction

Following [4], a numerical solution of the equation

\[ y(x) = \lambda \int_{a}^{b} K(x,t)y(t)dt = f(x), \quad a \leq x \leq b \]

with arbitrary kernel \( K(x,t) \) continuous in \( a \leq x, t \leq b \), is considered. Such a solution at points \( x_i \) of \( I = [a,b] \) is obtained from the system

\[ y_i^{(n)} - \lambda \sum_{j=1}^{n} w_i^{(n)} K_{ij} y_j^{(n)} = f_i, \quad i=1, \ldots, n \]

where \( K_{ij} = K(x_i,x_j) \), \( f_i = f(x_i) \), and \( w_i^{(n)} > 0 \) are coefficients of the integration method \( S \), the assumptions for \( K(x,t) \), \( f(x) \), \( \lambda \) and \( S \) being as in [4].

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In fact, let \( y^{(n)} \) be an approximate solution of (2) obtained by actual computation, and let

\[
\rho_i^{(n)} = y_i^{(n)} - \lambda \sum_{j=1}^{n} w_j^{(n)} K_{ij} y_j^{(n)} - f_i, \quad i = 1, 2, \ldots, n.
\]

Define the function

\[
y_n(x) = \lambda \sum_{j=1}^{n} w_j^{(n)} y_j^{(n)} K(x, x_j) + f(x)
\]

Then an estimate for the error \( e_n(x) = y_n(x) - y(x) \) is to be found.

The symmetric case with a special kind of quadrature formula (namely, such that the eigenvalues \( \mu_i^{(n)} \) of the matrices \( (w_j^{(n)} K_{ij}) \) converge to the reciprocal eigenvalues \( \lambda_i^{-1} \) according to Wielandt [6]) and other methods are considered in [4]. Another error estimate for the above method, which requires a bound for the maximum norm of the inverse operator in (1) or (2), is given by Anselone and Moore [1].

2. Error Bound

Introducing, in analogy to [4], the following notation:

\[
\varepsilon_n(x) = \sum_{i=1}^{n} w_i^{(n)} K(x, x_i) f(x_i) - \int_{a}^{b} K(x, t) f(t) dt
\]

\[
\eta_n(x, t) = \sum_{i=1}^{n} w_i^{(n)} K(x, x_i) K(x_i, t) - \int_{a}^{b} K(x, z) K(z, t) dz
\]

\[
a_n = \max_{I} |\varepsilon_n(x)|
\]

\[
A_n = \|\varepsilon_n\| \quad \text{where} \quad \|u\| = \left( \int_{a}^{b} u^2(x) dx \right)^{1/2}
\]

\[
M = \max_{I \times I} |K(x, t)|
\]
Then (see [4])

\[ e_n(x) = \lambda \int_a^b K(x,t)e_n(t)dt = \lambda E_n(x) \]

where

\[ E_n(x) = \lambda \sum_{i=1}^{n} w_i^{(n)} y_i^{(n)} \eta_n(x,x_i) + \epsilon_n(x) + \sum_{i=1}^{n} w_i^{(n)} \rho_i^{(n)} K(x,x_i) \]

Further

\[ |E_n(x)| \leq a_n + |\lambda| b_n + M c_n \]

\[ \|E_n\| \leq A_n + |\lambda| B_n + \sqrt{b-a} M c_n \]

where

\[ b_n = \sum_{i=1}^{n} w_i^{(n)} |y_i^{(n)}| \max_I |\eta_n(x,x_i)| \]

\[ B_n = \sum_{i=1}^{n} w_i^{(n)} |y_i^{(n)}| \left( \int_a^b \eta_n^2(x,x_i)dx \right)^{1/2} \]

\[ c_n = \sum_{i=1}^{n} w_i^{(n)} |\rho_i^{(n)}| \]

and the bounds for \( a_n, A_n, b_n, \) and \( B_n \) are evaluated from those of the error terms for \( \int_a^b K(x,t)f(t)dt \) and for \( \int_a^b K(x,z)K(z,t)dz \) in the method S.

By (3) and the Cauchy-Schwarz inequality

\[ |e_n(x)| \leq |\lambda| \left[ \|E_n(x)\| + \|e_n\| \sqrt{F(x)} \right] \]

where

\[ F(x) = \int_a^b K^2(x,t)dt \]
Now for every \( R(x,t) \in C(I \times I) \) and \( u(x) \in C(I) \) define the function \( v = Ru \) by

\[
v(x) = \int_a^b R(x,t)u(t)dt
\]

and let

\[
G(x,t) = \int_a^b K(z,x)K(z,t)dz
\]

It remains now to determine a constant \( \alpha > 0 \) such that for every \( u(x) \in C(I) \)

\[
\|u\| \leq \alpha \|u - \lambda Ku\|
\]

Then by (3)

\[
\|e_n\| \leq \alpha |\lambda| \|E_n\|
\]

and consequently, by (6)

\[
|e_n(x)| \leq |\lambda| \left[ |E_n(x)| + |\lambda| \alpha \|E_n\| \sqrt{f}\right]
\]

with bounds for \( E_n(x) \) and \( \|E_n\| \) given by (4) and (5).

For a symmetric kernel the value of \( \alpha \) in (8) was found in [4] to be

\[
\sup_i |1 - \lambda_i^{-1}|
\]

where \( \lambda_i, i=1,2,... \) are the eigenvalues. This result will be now generalized for every continuous kernel \( K(x,t) \).

From the definition of \( G(x,t) \) it follows that

\[
\|u - \lambda Ku\|^2 = \|u\|^2 - (Hu,u)
\]
where

\[ H(x,t) = \lambda [K(x,t) + K(t,x) - \lambda G(x,t)] \]

From the symmetry of \( H(x,t) \) it follows that either \( (Hu,u) \) is always \( \leq 0 \), or there exists

\[ A = \max_{\|u\|=1} (Hu,u) = (Hu_o,u_o) \]

with \( \|u_o\| = 1 \). Therefore, if \( \lambda \) is not an eigenvalue of \( K(x,t) \), then for every \( u(x) \) with \( \|u\| = 1 \)

\[ \|u - \lambda Ku\|^2 \geq 1 - (Hu_o,u_o) = 1 - A = \|u_o - \lambda Ku_o\|^2 > 0 \]

and (7) holds with \( \alpha = (1-A)^{-1/2} \).

3. Evaluation of \( \alpha \)

To find an upper bound for \( \alpha \), an upper bound \( C < 1 \) for \( A \) is required. Now, if the function \( G(x,t) \) defined above is unobtainable in exact form, then a symmetric approximation \( \tilde{G}(x,t) \) of \( G(x,t) \), having at least the same derivatives as \( G(x,t) \), can be found (e.g. by a quadrature formula). Further let

\[ \delta(x,t) = \tilde{G}(x,t) - G(x,t) \]

\[ \tilde{H}(x,t) = \lambda [K(x,t) + K(t,x) - \lambda \tilde{G}(x,t)] \]

Then

\[ \tilde{H}(x,t) = \tilde{H}(t,x) \]

\[ H(x,t) = \tilde{H}(x,t) + \lambda^2 \delta(x,t) \]
Now

\[ A = (H u_o, u_o) = (\tilde{H} u_o, u_o) + \lambda^2(\delta u_o, u_o) \leq \tilde{A} + \lambda^2(\delta u_o, u_o) \]

where

\[ \tilde{A} = \max_{\|u\|=1} (H u, u) \]

and by the Cauchy-Schwarz inequality

\[ (\delta u_o, u_o) \leq \gamma = \left[ \int_a^b \int_a^b \delta^2(x,t) dx dt \right]^{1/2} \]

Therefore

\[ (9) \quad A \leq \tilde{A} + \lambda^2 \gamma \]

and similarly

\[ (10) \quad \tilde{A} \leq A + \lambda^2 \gamma \]

Now choose \( \tilde{G}(x,t) \) in such a way that

\[ (11) \quad A + 2\lambda^2 T \leq B < 1 \]

where \( T \) is a computed bound for \( \gamma \); then by (10)

\[ \tilde{A} \leq c < 1 - \lambda^2 T \]

and the required bound \( C = c + \lambda^2 T \) is obtained by (9).

To find an estimate for \( \tilde{A} \), the following result of Wielandt [6], which holds for some classes of quadrature formulae (depending on \( \tilde{H}(x,t) \)), may be used:
Let \( \tilde{\mu}_i^{(n)} \) be the sequences of eigenvalues of the matrices \( \tilde{H}^{(n)} \), where

\[
\tilde{H}^{(n)}_{ij} = \sqrt{w_i^{(n)} w_j^{(n)}} \tilde{H}(x_i, x_j), \quad i, j = 1, \ldots, n
\]

completed by \( \tilde{\mu}_i^{(n)} = 0 \) for \( i > n \), and let \( \tilde{\lambda}_i \) be the corresponding eigenvalues of \( \tilde{H}(x,t) \); then \( \tilde{\mu}_i^{(n)} \rightarrow \tilde{\lambda}_i^{-1} \) uniformly in \( i \), i.e.

\[
|\tilde{\mu}_i^{(n)} - \tilde{\lambda}_i^{-1}| < q_n \quad \text{where} \quad \lim_{n \to \infty} q_n = 0
\]

In particular

\[
\tilde{A} = \lim_{n \to \infty} \tilde{A}_n
\]

where \( \tilde{A}_n = \max_{|z|=1} z^* \tilde{H}^{(n)} z \) is the maximal eigenvalue of \( \tilde{H}^{(n)} \).

Now suppose that (11) holds; then there exists an integer \( N \) such that for every \( n > N \) and for a suitable choice of \( c \)

\[
(12) \quad \tilde{A}_n + q_n < c < 1 - \lambda^2 r
\]

and consequently

\[
\tilde{A} < \tilde{A}_n + |\tilde{A}_n - \tilde{A}| < \tilde{A}_n + q_n < c
\]

Therefore it remains to find a value of \( n \) (on assumption that (11) is satisfied) such that (12) holds.

4. **Convergence**

By the property of the quadrature formula

\[
(13) \quad \lim_{n \to \infty} a_n = \lim_{n \to \infty} \max_{x, t} |\eta_n(x, t)| = 0
\]

\[
(14) \quad \lim_{n \to \infty} \sum_{i=1}^{n} w_i^{(n)} = b - a
\]
If $\tilde{y}^{(n)}$ is the exact solution of (2), then $p_i^{(n)} = 0$ for each $i$, and consequently
\[ c_n = \sum_{i=1}^{n} w_i^{(n)} |p_i^{(n)}| = 0. \]

Now introduce the Chebycheff norm $\| \|$; then, to obtain a uniform convergence of $y_n(x)$ to $y(x)$, it suffices to show that the sequence $\| \tilde{y}^{(n)} \|$ is bounded. Indeed
\[ |E_n(x)| \leq |\lambda| \sum_{i=1}^{n} w_i^{(n)} |\tilde{y}_i^{(n)}| |n(x,x_i)| + |e_n(x)| \leq \]
\[ \leq |\lambda| \| \tilde{y}^{(n)} \| \max_{I \times I} |n(x,t)| \sum_{i=1}^{n} w_i^{(n)} + a_n \]
and the required result follows by (13), (14) and (8).

A topological proof of convergence is given in [1]. Another proof, by purely analytical methods, will be given below.

Let $P_n$ be the matrix of the system (2), $R(x,t,\lambda)$ the resolvent of the equation (1), and let
\[ N = \max_{I \times I} |R(x,t,\lambda)|. \]
Then
\[ P_n = I - \lambda L^{(n)} \]
where
\[ L^{(n)} = w^{(n)} k \]
and it is to be proved that the sequence $\| P_n^{-1} \|$ is bounded. For this purpose we show that (see [3], where the formula contains a minor misprint)
\[ P_n^{-1} = (I - \lambda^2 G^{(n)})^{-1}(I + \lambda R^{(n)}) \]
where
\[ G_{ij}^{(n)} = \sum_j w_j \left[ \lambda \int_a^b R(x_i, t, \lambda) n(t, x_j) dt + n(x_i, x_j) \right] \]

\[ R_{ij}^{(n)} = \sum_j w_j R(x_i, x_j, \lambda) \]

Then by (13) and (14) it follows that
\[ \lim_{n \to \infty} \| R^{(n)} \| < (b-a)N \]
\[ \lim_{n \to \infty} \| G^{(n)} \| = 0 \]

and consequently
\[ \lim_{n \to \infty} \left\| P_n^{-1} \right\| \leq [1 + |\lambda|(b-a)N] \lim_{n \to \infty} (1 - \lambda^2 \| G^{(n)} \|)^{-1} = 1 + |\lambda|(b-a)N \]

i.e. \( \| P_n^{-1} \| \) is bounded.

Proof of (15): The resolvent \( R(x, t, \lambda) \) satisfies the equation
\[ R(x, t, \lambda) - \lambda \int_a^b R(x, z, \lambda) K(z, t) dz = K(x, t) \]

Then
\[ \lambda \int_a^b R(x_i, t, \lambda) n(t, x_j) dt = \lambda \int_a^b R(x_i, t, \lambda) \left[ \sum_{m=1}^n w_m K(t, x_m) K_{mj} - \lambda \int_a^b K(t, z) K(z, x_j) dz \right] dt - \lambda \int_a^b \left[ \int_a^b R(x_i, z, \lambda) K(t, z) dz \right] K(z, x_j) dz \]

\[ - \lambda \int_a^b R(x_i, z, \lambda) K(t, z) dz \]

\[ - \int_a^b R(x_i, z, \lambda) K(z, x_j) dz = \sum_{m=1}^n w_m [R(x_i, x_m, \lambda) - K_{im}] K_{mj} \]

\[ - \int_a^b [R(x_i, z, \lambda) - K(x_i, z)] K(z, x_j) dz \]

\[ - \int_a^b R(x_i, z, \lambda) K(z, x_j) dz - n(x_i, x_j) \]
Hence

\[ G_{ij}^{(n)} = w_{ij}^{(n)} \left[ \sum_{m=1}^{n} w_{mj}^{(n)} R(x_i, x_m, \lambda) K_m - \int_{a}^{b} R(x_i, z, \lambda) K(z, x_j) dz \right] \]

Now

\[ w_{ij}^{(n)} \sum_{m=1}^{n} w_{mj}^{(n)} R(x_i, x_m, \lambda) K_m = H_{ij}^{(n)} \]

where \( H^{(n)} = R(n)L(n) \), and by (16)

\[ \lambda w_{ij}^{(n)} \int_{a}^{b} R(x_i, z, \lambda) K(z, x_j) dz = R_{ij}^{(n)} - L_{ij}^{(n)} \]

Therefore

\[ I - \lambda^2 G^{(n)} = I - \lambda^2 R(n)L(n) + \lambda (R(n) - L(n)) = (I + \lambda R(n))(I - \lambda L(n)) \]

which proves (15).

5. Numerical Results

In the following examples all the numerical results are computed at points with constant distance \( h \), including the endpoints. The new estimates are then compared with those of Kantorovich and Krylov [2].

Equation

(17) \[ y(x) + \int_{0}^{1} xe^{xt} y(t) dt = e^x \]

Solution

\[ y(x) \equiv 1 \]
Bounds

\[ M = e \]

\[ F(x) = \int_0^1 x^2 e^{2xt} dt = \frac{x}{2} (e^{2x} - 1) < \frac{1}{2} (e^2 - 1) = 3.1945 \ldots \]

A bound for \( \alpha \), obtained by taking

\[ \tilde{G}(x,t) = \sum_{n=0}^{N} \frac{(x+t)^n}{n!(n+3)} \]

and using results of Wielandt [6] for Simpson's quadrature formula, is

\[ \alpha < 1.021 \]

Denote

\[ C_n = [180(n-1)^4]^{-1} \]

\[ u(x,t) = K(x,t)f(t) = xe^{(x+1)t} \quad v(x,t,z) = K(x,z)K(z,t) = xze^{(x+t)z} \]

Then

\[ 0 \leq \epsilon_n(x) \leq C_n \left[ \frac{\partial^3 u}{\partial t^3} (x,1) - \frac{\partial^3 u}{\partial t^3} (x,0) \right] = C_n x(x+1)^3 (e^{x+1} - 1) \]

\[ 0 \leq \eta_n(x,t) \leq C_n \left[ \frac{\partial^3 v}{\partial z^3} (x,t,1) - \frac{\partial^3 v}{\partial z^3} (x,t,0) \right] = \]

\[ = C_n x(x+t)^2 [(x+t+3) e^{x+t} - 3] \leq C_n (1+t)^2 [(t+4) e^{t+1} - 3] \]

since

\[ \frac{\partial^k u}{\partial t^k} (x,t) \geq 0, \quad \frac{\partial^k v}{\partial z^k} (x,t,z) \geq 0, \quad k=1,2,\ldots \]
for every \( 0 \leq x, t, z \leq 1 \). Consequently

\[
A_n < C_n \left[ \int_0^1 x(x+1)^6 (e^{x+1} - 1)^2 \, dx \right]^{1/2} < C_n \left[ \sum_{i=1}^n w_i \frac{x}{x_i+1} (x_i+1)^6 (e^{x_i+1} - 1)^2 \right]^{1/2}
\]

and similarly

\[
\int_0^2 \frac{x^2(x,t)^2}{n} \, dx < C_n \left[ \sum_{i=1}^n w_i \frac{x}{x_i(x_i+t)^4} (x_i+t+3)e^{x_i+t-3} \right]^2
\]

Case 1: \( n = 11 \)

\[
a_n < 2.84 \cdot 10^{-5} \quad A_n < 1.02 \cdot 10^{-5}
\]

\[
b_n < 2.75 \cdot 10^{-5} \quad B_n < 8.32 \cdot 10^{-6} \quad c_n < 7.1 \cdot 10^{-9}
\]

Error estimate

\[
|e_n(x)| < 8.97 \cdot 10^{-5}
\]

The actual maximal error at \( x = 0.02j \) is about \( 7.5 \cdot 10^{-7} \).

Using the notations of Kantorovich and Krylov [2] (p.103-107) the following results are obtained:

\[
s = 4, \quad k_n = C_n
\]

(18)

\[
S \leq \max_i |y_i^{(n)}| + \frac{1}{2n-2} \max_i |y_i'(x)|
\]

(19)

\[
N_o = N(x) = N_t = M_r(x) = e, \quad N_r(x) = (r+1)e
\]

(20)

\[
P_s = 2^s e^2, \quad Q_s = (1 + \frac{s}{2})2^s e^2
\]
Now

$$B = 1.806$$

and by (17)

$$S < 1.14$$

Hence ([2] p.107)

$$H(0) < 1.15$$

and consequently ([2] p.106)

$$|e_n(x)| < 1.73 \cdot 10^{-3}$$

which is about 19 times greater than our estimate.

Case 2: \( n = 21 \)

\[
\begin{align*}
a_n &< 1.78 \cdot 10^{-6} \\
A_n &< 6.3 \cdot 10^{-7} \\
b_n &< 1.72 \cdot 10^{-6} \\
B_n &< 5.2 \cdot 10^{-7} \\
c_n &< 7 \cdot 10^{-9}
\end{align*}
\]

Error estimate

$$|e_n(x)| < 5.64 \cdot 10^{-6}$$

The actual maximal error at \( x = 0.02j \) is about \( 5 \cdot 10^{-8} \).

Now

$$B = 1.857$$

and by (18)

$$S < 1.068$$

which together with (19) and (20) yields in a similar manner

$$|e_n(x)| < 1.05 \cdot 10^{-4}$$
Numerical solution of equation (17) by the Gaussian quadrature yields the following results:

By [5] p. 48

\[ e_n(x) = -k_n \frac{2^n}{2^n x} [xe(x+t)t] = -k_n x(x+1)^{2n} e(x+1)t \quad 0 < t = \tau(x) < 1 \]

\[ \eta_n(x,t) = -k_n \frac{2^n}{2^n z} [xze(x+t)z] = -k_n x(x+t)^{2n-1} [\xi(x+t)+2n]e(x+t)\xi \quad 0 < \xi = \xi(x,t) < 1 \]

where

\[ k_n = \left[ \binom{2n}{n}^2 (2n+1)! \right]^{-1} \]

Thus

\[ a_n < 2^n k \frac{e^2}{n} \quad A_n < k e^1 \frac{1}{\int_0^1 x^2(x+1)^{4n} e^{2x} dx} \frac{1}{\sqrt{2}} \]

(21) \[ |\eta_n(x,t)| < k_n x(x+t)^{2n-1} (x+t+2n)e^{x+t} \]

By the notations of [2] p.106

\[ \sigma \leq k_n \tau(2n) \leq k_n (p_{2n} + Q_{2n}) \]

where \( p_{2n} \) and \( Q_{2n} \) are defined by (20).

For \( n = 4 \)

\[ a_4 < 1.06 \cdot 10^{-6} \quad A_4 < 3.2 \cdot 10^{-7} \quad c_4 < 4.5 \cdot 10^{-9} \]

and by (21)

\[ b_4 < 1.063 \cdot 10^{-6} \quad B_4 < 3 \cdot 10^{-7} \]

Hence

\[ |e_n(x)| < 3.273 \cdot 10^{-6} \]
The actual maximal error at $x = 0.02j$ is about $2 \cdot 10^{-8}$.

Using the notations of [2] the following results are obtained:

$$B = 1.387$$

$$S \leq \max |y_1^{(4)}| + \max(d_i |0 \leq i \leq 4| \max |y_4'(x)| < 1.462$$

where

$$d_0 = x_1, \quad d_4 = 1-x_4, \quad d_i = \frac{1}{2} (x_{i+1} - x_i), \quad i=1,2,3$$

Hence

$$H^{(0)} < 1.463$$

and consequently

$$|e_n(x)| < 4.22 \cdot 10^{-5}$$

REFERENCES


