Efficient Craig Interpolation for Linear Diophantine (Dis)Equations and Linear Modular Equations *

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Abstract. The use of Craig interpolants has enabled the development of powerful hardware and software model checking techniques. Efficient algorithms are known for computing interpolants in rational and real linear arithmetic. We focus on subsets of integer linear arithmetic. Our main results are polynomial time algorithms for obtaining interpolants for conjunctions of linear diophantine equations, linear modular equations (linear congruences), and linear diophantine disequations. We show the utility of the proposed interpolation algorithms for discovering *modular/divisibility* predicates in a counterexample guided abstraction refinement (CEGAR) framework. This has enabled verification of simple programs that cannot be checked using existing CEGAR based model checkers.

1 Introduction

The use of Craig interpolation [8] has led to powerful hardware [14] and software [9] model checking techniques. In [14] the idea of interpolation is used for obtaining over-approximations of the reachable set of states without using the costly image computation (existential quantification) operations. In [9, 11] interpolants are used for finding the *right* set of predicates in order to rule out *spurious counterexamples*. An interpolating theorem prover performs the task of finding the interpolants. Such provers are available for various theories such as propositional logic, rational and real linear arithmetic, and equality with uninterpreted functions [15, 21, 12, 11, 19, 13, 6].

Efficient algorithms are known for computing interpolants in rational and real linear arithmetic [15, 19, 6]. Linear arithmetic formulas where all variables are constrained to be integers are said to be formulas in *(pure) integer linear arithmetic* or $LA(\mathbb{Z})$, where $\mathbb{Z}$ is the set of integers. There are no known efficient algorithms for computing interpolants for formulas in $LA(\mathbb{Z})$. This is expected because checking the satisfiability of conjunctions of atomic formulas in $LA(\mathbb{Z})$ is itself NP-hard. We show that for various *subsets* of $LA(\mathbb{Z})$ one can compute interpolants efficiently.

Informally, a linear equation where all variables are integer variables is said to be a *linear diophantine equation (LDE)*. A *linear modular equation (LME)* or a *linear congruence* over integer variables is a type of linear equation that expresses divisibility relationships. A *system* of LDEs (LMEs) denotes conjunctions of LDEs (LMEs). Both LDEs and LMEs arise naturally in program verification when modeling assignments and conditional statements as logical formulas. These subsets of $LA(\mathbb{Z})$ are also known

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to be tractable, that is, polynomial time algorithms are known for deciding systems of LDEs and LMEs. We study the interpolation problem for LDEs and LMEs.

Given formulas F, G such that $F \wedge G$ is unsatisfiable, an interpolant for the pair (F, G) is a formula $I(F, G)$ with the following properties: 1) F implies $I(F, G)$, 2) $I(F, G) \wedge G$ is unsatisfiable, and 3) $I(F, G)$ refers only to the common variables of F and G. This paper presents the following new results.

- F, G denote a system of LDEs: We show that $I(F, G)$ can be obtained in polynomial time by using a proof of unsatisfiability of $F \wedge G$. The interpolant can be either a LDE or a LME. This is because in some cases there is no $I(F, G)$ that is a LDE. In these cases, however, there is always an $I(F, G)$ in the form of a LME. (Sec. 3)
- F, G denote a system of LMEs: We obtain $I(F, G)$ in polynomial time by using a proof of unsatisfiability of $F \wedge G$. We can ensure that $I(F, G)$ is a LME. (Sec. 4)
- Let S denote an unsatisfiable system of LDEs. The proof of unsatisfiability of S can be obtained in polynomial time by using the *Hermite Normal Form* of S (represented in matrix form) [20]. A system of LMEs R can be reduced to an equisatisfiable system of LDEs R'. The proof of unsatisfiability for R is easily obtained from the proof of unsatisfiability of R'. (Sec. 5)
- Let S denote a system of LDEs. We show that if S has an integral solution, then every LDE that is implied by S, can be obtained by a linear combination of equations in S. We show that S is *convex* [17], that is, if S implies a disjunction of LDEs, then it implies one of the equations in the disjunction. In contrast, conjunctions of atomic formulas in $LA(\mathbb{Z})$ are not convex due to inequalities [17]. These results help in efficiently dealing with *linear diophantine disequations (LDDs)*. (Sec. 6)
- Let $S = S_1 \wedge S_2$, where S_1 is a system of LDEs, while S_2 is a system of LDDs. We say that S is a system of LDEs+LDDs. We show that S has no integral solution if and only if $S_1 \wedge S_2$ has no rational solution or S_1 has no integral solution. This gives a polynomial time decision procedure for checking if S has an integral solution. If S has no integral solution, then the proof of unsatisfiability of S can be obtained in polynomial time. (Sec. 6)
- F, G denote a system of LDEs+LDDs: We show $I(F, G)$ can be obtained in polynomial time. The interpolant can be a LDE, a LDD, or a LME. (Sec. 6)
- We show the utility of our interpolation algorithms in counterexample guided abstraction refinement (CEGAR) based verification [7]. Our interpolation algorithm is effective at discovering *modular/divisibility predicates*, such as $3x + y + 2z \equiv 1 \ (mod\, 4)$, from spurious counterexamples. This has allowed us to verify programs that cannot be verified by existing CEGAR based model checkers.

Polynomial time algorithms are known for solving (deciding) a system of LDEs [20, 5] and LMEs (by reduction to LDEs) over integers. We do not give any new algorithms for solving a system of LDEs or LMEs. Instead we focus on obtaining proofs of unsatisfiability and interpolants for systems of LDEs, LMEs, LDEs+LDDs. We only consider conjunctions of LDEs, LMEs, LDEs+LDDs. Interpolants for any (unsatisfiable) Boolean combinations of LDEs can also be obtained by calling the interpolation algorithm for conjunctions of LDEs+LDDs multiple times in a satisfiability modulo theory

(SMT) framework [6]. However, computing interpolants for Boolean combinations of LMEs is difficult. This is due to linear modular disequations (LMDs). We can show that even the decision problem for conjunctions of LMDs is NP-hard. The extended version of the paper [10] contains all proofs.

Related Work. It is known that Presburger arithmetic (PA) augmented with modulus operator (divisibility predicates) allows quantifier elimination. Kapur et al. [12] show that a recursively enumerable theory allows quantifier-free interpolants if and only if it allows quantifier elimination. The systems of LDEs, LMEs, LDEs+LDDs are subsets of PA. Thus, the existence of quantifier-free interpolants for these systems follows from [12]. However, quantifier elimination for PA has exponential complexity and does not immediately yield efficient algorithms for computing interpolants. We give polynomial time algorithms for computing interpolants for systems of LDEs, LMEs, LDEs+LDDs.

Let S_1, S_2 denote conjunctions of atomic formulas in $LA(\mathbb{Z})$. Suppose $S_1 \wedge S_2$ is unsatisfiable. Pudlak [18] shows how to compute an interpolant for (S_1, S_2) by using a *cutting-plane* (CP) proof of unsatisfiability. The CP proof system is a sound and complete way of proving unsatisfiability of conjunctions of atomic formulas in $LA(\mathbb{Z})$. However, a CP proof for a formula can be exponential in the size of the formula. Pudlak does not provide any guarantee on the size of CP proofs for a system of LDEs or LMEs. Our results show that polynomially sized proofs of unsatisfiability and interpolants can be obtained for systems of LDEs, LMEs and LDEs+LDDs.

McMillan [15] shows how to compute interpolants in the combined theory of rational linear arithmetic $LA(\mathbb{Q})$ and equality with uninterpreted functions $\mathcal{EUF}$ by using proofs of unsatisfiability. Rybalchenko and Sofronie-Stokkermans [19] show how to compute interpolants in combined $LA(\mathbb{Q})$, $\mathcal{EUF}$ and real linear arithmetic $LA(\mathbb{R})$ by using linear programming solvers in a black-box fashion. The key idea in [19] is to use an extension of Farkas lemma [20] to reduce the interpolation problem to constraint solving in $LA(\mathbb{Q})$ and $LA(\mathbb{R})$. Cimatti et al. [6] show how to compute interpolants in a satisfiability modulo theory (SMT) framework for $LA(\mathbb{Q})$, rational difference logic fragment and $\mathcal{EUF}$. By making use of state-of-the-art SMT algorithms they obtain significant improvements over existing interpolation tools for $LA(\mathbb{Q})$ and $\mathcal{EUF}$. Yorsh and Musuvathi [21] give a Nelson-Oppen [17] style method for generating interpolants in a combined theory by using the interpolation procedures for individual theories. Kroening and Weissenbacher [13] show how a bit-level proof can be lifted to a word-level proof of unsatisfiability (and interpolants) for equality logic.

To the best of our knowledge the work in [15, 21, 19, 13, 6] is not complete for computing interpolants in $LA(\mathbb{Z})$ or its subsets such as LDEs, LMEs, LDEs+LDDs. That is, the work in [15, 21, 19, 13, 6] cannot compute interpolants for formulas that are satisfiable over rationals but unsatisfiable over integers. Such formulas can arise in both hardware and software verification. We give sound and complete polynomial time algorithms for computing interpolants for conjunctions of LDEs, LMEs, LDEs+LDDs.

2 Notation and Preliminaries

We use capital letters $A, B, C, X, Y, Z, \ldots$ to denote matrices and formulas. A matrix M is *integral (rational)* iff all elements of M are integers (rationals). For a matrix

M with m rows and n columns we say that the size of M is $m \times n$. A *row vector* is a matrix with a single row. A *column vector* is a matrix with a single column. We sometimes identify a matrix M of size 1×1 by its only element. If A, B are matrices, then AB denotes matrix multiplication. We assume that all matrix operations are well defined in this paper. For example, when we write AB without specifying the sizes of matrices A, B, it is assumed that the number of columns in A equals the number of rows in B.

For any rational numbers α and β, $\alpha|\beta$ if and only if, α divides β, that is, if and only if $\beta = \lambda\alpha$ for some integer λ. We say that α is equivalent to β *modulo* γ written as $\alpha \equiv \beta \ (mod\ \gamma)$ if and only if $\gamma|(\alpha - \beta)$. We say γ is the *modulus* of the equation $\alpha \equiv \beta \ (mod\ \gamma)$. We allow α, β, γ to be rational numbers. If $\alpha_1, \ldots, \alpha_n$ are rational numbers, not all equal to 0, then the largest rational number γ dividing each of $\alpha_1, \ldots, \alpha_n$ exists [20], and is called the *greatest common divisor*, or *gcd* of $\alpha_1, \ldots, \alpha_n$ denoted by $gcd(\alpha_1, \ldots, \alpha_n)$. We assume that gcd is always positive.

Basic Properties of Modular Arithmetic: Let a, b, c, d, m be rational numbers.
P1. $a \equiv a \ (mod\ m)$ (reflexivity).
P2. $a \equiv b \ (mod\ m)$ implies $b \equiv a \ (mod\ m)$ (symmetry).
P3. $a \equiv b \ (mod\ m)$ and $b \equiv c \ (mod\ m)$ imply $a \equiv c \ (mod\ m)$ (transitivity).
P4. If $a \equiv b \ (mod\ m)$, $c \equiv d \ (mod\ m)$, and x, y are integers, then $ax + cy \equiv bx + dy \ (mod\ m)$ (integer linear combination).
P5. If $c > 0$ then $a \equiv b \ (mod\ m)$ if, and only if, $ac \equiv bc \ (mod\ mc)$.
P6. If $a = b$, then $a \equiv b \ (mod\ m)$ for any m.

Example 1. Observe that $x \equiv 0 \ (mod\ 1)$ for any integer x. Also observe from P5 (with $c = 2$) that $\frac{1}{2}x \equiv 0 \ (mod\ 1)$ if and only if $x \equiv 0 \ (mod\ 2)$.

A *linear diophantine equation (LDE)* is a linear equation $c_1x_1 + \ldots + c_nx_n = c_0$, where $x_1, \ldots, x_n$ are integer variables and $c_0, \ldots, c_n$ are rational numbers. A variable x_i is said to *occur* in the LDE if $c_i \neq 0$. We denote a system of m LDEs in a matrix form as $CX = D$, where C denotes an $m \times n$ matrix of rationals, X denotes a column vector of n integer variables and D denotes a column vector of m rationals. When we write a (single) LDE in the form $CX = D$, it is implicitly assumed that the sizes of C, X, D are of the form $1 \times n, n \times 1, 1 \times 1$, respectively. A variable is said to *occur* in a system of LDEs if it occurs in at least one of the LDEs in the given system of LDEs.

A *linear modular equation (LME)* has the form $c_1x_1 + \ldots + c_nx_n \equiv c_0 \ (mod\ l)$, where $x_1, \ldots, x_n$ are integer variables, $c_0, \ldots, c_n$ are rational numbers, and l is a rational number. We call l the modulus of the LME. Allowing l to be a rational number leads to simpler proofs and covers the case when l is an integer. We abbreviate a LME $t \equiv c \ (mod\ l)$ by $t \equiv_l c$. A variable x_i is said to *occur* in a LME if l does not divide c_i.

A *system* of LDEs (LMEs) denotes conjunctions of LDEs (LMEs). If F, G are a system of LDEs (LMEs), then $F \wedge G$ is also a system of LDEs (LMEs).

2.1 Craig Interpolants

Given two logical formulas F and G in a theory $\mathcal{T}$ such that $F \wedge G$ is unsatisfiable in $\mathcal{T}$, an interpolant I for the ordered pair (F, G) is a formula such that
(1) $F \Rightarrow I$ in $\mathcal{T}$

(2) $I \wedge G$ is unsatisfiable in $\mathcal{T}$
(3) I refers to only the common variables of A and B.
The interpolant I can contain symbols that are interpreted by $\mathcal{T}$. In this paper such symbols will be one of the following: addition ($+$), equality ($=$), modular equality for some rational number m ($\equiv_m$), disequality ($\neq$), and multiplication by a rational number ($\times$). The exact set of interpreted symbols in the interpolant depends on $\mathcal{T}$.

3 System of Linear Diophantine Equations (LDEs)

In this section we discuss proofs of unsatisfiability and interpolation algorithm for LDEs. The following theorem from [20] gives a necessary and sufficient condition for a system of LDEs to have an integral solution.

Theorem 1. *(Corollary 4.1(a) in Schrijver [20]) A system of LDEs $CX = D$ has no integral solution for X, if and only if there exists a rational row vector R such that RC is integral and RD is not an integer.*

Definition 1. *We say a system of LDEs $CX = D$ is* **unsatisfiable** *if it has no integral solution for X. For a system of LDEs $CX = D$ a* **proof of unsatisfiability** *is a rational row vector R such that RC is integral and RD is not an integer.*

Example 2. Consider the system of LDEs $CX = D$ and a proof of unsatisfiability R:

$$CX = D := \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \qquad \begin{array}{l} R \;\;\, = [\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}] \\ RC = [0, 2, 1] \\ RD = \frac{3}{2} \end{array}$$

Example 3. Consider the system of LDEs $CX = D$ and a proof of unsatisfiability R:

$$CX = D := \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{array}{l} R \;\;\, = [\frac{1}{2}, \frac{1}{2}] \\ RC = [1, -1, -1] \\ RD = \frac{1}{2} \end{array}$$

The above examples will be used as running examples in the paper. In section 5 we describe how a proof of unsatisfiability can be obtained in polynomial time for an unsatisfiable system of LDEs.

Definition 2. *(***Implication***) A system of LDEs $CX = D$* **implies** *a (single) LDE $AX = B$, if every integral vector X satisfying $CX = D$ also satisfies $AX = B$.*

Similarly, $CX = D$ **implies** *a (single) LME $AX \equiv_m B$, if every integral vector X satisfying $CX = D$ also satisfies $AX \equiv_m B$.*

Lemma 1. *(Linear combination) For every rational row vector U the system of LDEs $CX = D$ implies the LDE $UCX = UD$. Note that $UCX = UD$ is simply a linear combination of the equations in $CX = D$. The system $CX = D$ also implies the LME $UCX \equiv_m UD$ for any rational number m.*

Example 4. The system of LDEs $CX = D$ in Example 3 implies the LDE $[\frac{1}{2}, \frac{1}{2}]CX = [\frac{1}{2}, \frac{1}{2}]D$, which simplifies to $x - y - z = \frac{1}{2}$. The system $CX = D$ also implies the LME $x - y - z \equiv_m \frac{1}{2}$ for any rational number m.

3.1 Computing Interpolants for Systems of LDEs

Let $F \wedge G$ denote an unsatisfiable system of LDEs. The following example shows that an unsatisfiable system of LDEs does not always have a LDE as an interpolant.

Example 5. Let $F := x - 2y = 0$ and $G := x - 2z = 1$. Intuitively, F expresses the constraint that x is even and G expresses the constraint that x is odd, thus, $F \wedge G$ is unsatisfiable. We gave a proof of unsatisfiability of $F \wedge G$ in Example 3. Observe that the pair (F, G) does not have any quantifier-free interpolant that is also a LDE. The problem is that the interpolant can only refer to the variable x. We can show that there is no formula I of the form $c_1 x + c_2 = 0$, where c_1, c_2 are rational numbers, such that $F \Rightarrow I$ and $I \wedge G$ is unsatisfiable (see [10] for proof).

As shown by the above example it is possible that there exists no LDE that is an interpolant for (F, G). We show that in this case the system (F, G) always has a LME as an interpolant. In the above example an interpolant will be $x \equiv_2 0$. Intuitively, the interpolant means that x is an even integer.

We now describe the algorithm for obtaining interpolants. Let $AX = A', BX = B'$ be systems of LDEs, where $X = [x_1, \ldots, x_n]$ is a column vector of n integer variables. Suppose the combined system of LDEs $AX = A' \wedge BX = B'$ is unsatisfiable. We want to compute an interpolant for $(AX = A', BX = B')$. Let $R = [R_1, R_2]$ be a proof of unsatisfiability of $AX = A' \wedge BX = B'$ such that

$$R_1 A + R_2 B \quad \text{is integral and} \quad R_1 A' + R_2 B' \quad \text{is not an integer.}$$

Recall that a variable is said to *occur* in a system of LDEs if it occurs with a non-zero coefficient in one of the equations in the given system of LDEs. Let $V_{AB} \subseteq X$ denote the set of variables that occur in both $AX = A'$ and $BX = B'$, let $V_{A \setminus B} \subseteq X$ denote the set of variables occurring only in $AX = A'$ (and not in $BX = B'$), and let $V_{B \setminus A} \subseteq X$ denote the set of variables occurring only in $BX = B'$ (and not in $AX = A'$).

We call the LDE $R_1 AX = R_1 A'$ a **partial interpolant** for $(AX = A', BX = B')$. It is a linear combination of equations in $AX = A'$. The partial interpolant $R_1 AX = R_1 A'$ can be written in the following form

$$\sum_{x_i \in V_{A \setminus B}} a_i x_i + \sum_{x_i \in V_{AB}} b_i x_i = c \tag{1}$$

where all coefficients a_i, b_i and $c = R_1 A'$ are rational numbers. Observe that the partial interpolant does not contain any variable that occurs only in $BX = B'$ ($V_{B \setminus A}$).

Lemma 2. *The coefficient a_i of each $x_i \in V_{A \setminus B}$ in the partial interpolant $R_1 AX = R_1 A'$ (Equation 1) is an integer.*

Lemma 3. *The partial interpolant $R_1 AX = R_1 A'$ satisfies the first two conditions in the definition of an interpolant. That is,*
1. $AX = A'$ implies $R_1 AX = R_1 A'$
2. $(R_1 AX = R_1 A') \wedge BX = B'$ is unsatisfiable
If $a_i = 0$ for all $x_i \in V_{A \setminus B}$ (equation 1), then the partial interpolant only contains the variables from V_{AB}. In this case the partial interpolant is an interpolant for $(AX = A', BX = B')$. (The proof is given in [10].)

Example 6. Consider the system of LDEs $CX = D$ in Example 2. A proof of unsatisfiability for this system is $R = [\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]$. Let $AX = A'$ be the first two equations in $CX = D$, that is, $x + y = 1 \wedge x - y = 1$ (in matrix form). Let $BX = B'$ be the third equation in $CX = D$, that is, $2y + 2z = 3$. Observe that $V_{A\setminus B} := \{x\}, V_{AB} := \{y\}, V_{B\setminus A} := \{z\}$. In this case $R_1 = [\frac{1}{2}, -\frac{1}{2}]$. The partial interpolant for the pair $(AX = A', BX = B')$ is $y = 0$, which is also an interpolant because $y \in V_{AB}$.

The following example shows that a partial interpolant need not be an interpolant.

Example 7. Consider the system $CX = D$ in Example 3. A proof of unsatisfiability for this system is $R = [\frac{1}{2}, \frac{1}{2}]$. Let $AX = A'$ be the first equation in $CX = D$, that is, $x - 2y = 0$. Let $BX = B'$ be the second equation in $CX = D$, that is, $x - 2z = 1$. Observe that $V_{A\setminus B} := \{y\}, V_{AB} := \{x\}, V_{B\setminus A} := \{z\}$. In this case $R_1 = [\frac{1}{2}]$. Thus, the partial interpolant for the pair $(AX = A', BX = B')$ is $\frac{1}{2}x - y = 0$. Observe that the partial interpolant is not an interpolant as it contains the variable y, which does not occur in V_{AB}. This is not surprising since we have already seen in Example 5 that $(x - 2y = 0, x - 2z = 1)$ cannot have an interpolant that is a LDE.

We now intuitively describe how to remove variables from the partial interpolant that are not common to $AX = A'$ and $BX = B'$. In example 7 the partial interpolant is $\frac{1}{2}x - y = 0$, where $y \notin V_{AB}$. We show how to eliminate y from $\frac{1}{2}x - y = 0$ in order to obtain an interpolant. We use modular arithmetic in order to eliminate y. Informally, the equation $\frac{1}{2}x - y = 0$ implies $\frac{1}{2}x - y \equiv 0 \ (mod\ \gamma)$ for any rational number γ. Let α denote the greatest common divisor of the coefficients of variables (in $\frac{1}{2}x - y = 0$) that do not occur in V_{AB}. In this example $\alpha = 1$ (gcd of the coefficient of y). We know $\frac{1}{2}x - y = 0$ implies $\frac{1}{2}x - y \equiv 0 \ (mod\ 1)$. Since y is an integer variable $y \equiv 0 \ (mod\ 1)$. We can add $\frac{1}{2}x - y \equiv 0 \ (mod\ 1)$ and $y \equiv 0 \ (mod\ 1)$ to obtain $\frac{1}{2}x \equiv 0 \ (mod\ 1)$ (note that y is eliminated). Intuitively, the linear modular equation $\frac{1}{2}x \equiv 0 \ (mod\ 1)$ is an interpolant for $(x - 2y = 0, x - 2z = 1)$. By using basic modular arithmetic this interpolant can be written as $x \equiv 0 \ (mod\ 2)$.

We now formalize the above intuition to address the case when the partial interpolant contains variables that are not common to $AX = A'$ and $BX = B'$.

Theorem 2. *Assume that the coefficient a_i of at least one $x_i \in V_{A\setminus B}$ in the partial interpolant (Equation 1) is not zero. Let α denote the gcd of $\{a_i | x_i \in V_{A\setminus B}\}$.*
(a) α is an integer and $\alpha > 0$.
(b) Let β be any integer that divides α. Then the following linear modular equation I_β is an interpolant for $(AX = A', BX = B')$.

$$I_\beta := \sum_{x_i \in V_{AB}} b_i x_i \equiv c \ (mod\ \beta)$$

Observe that I_β contains only variables that are common to both $AX = A'$ and $BX = B'$. It is obtained from the partial interpolant by dropping all variables occurring only in $AX = A'$ ($V_{A\setminus B}$) and replacing the linear equality by a modular equality.

The complete proof can be found in [10]. Lemma 3 and Theorem 2 give us a sound and complete algorithm for computing an interpolant for unsatisfiable systems of LDEs (see [10] for algorithm pseudocode). In theorem 2, I_1 is always an interpolant for $(AX = A', BX = B')$. For $\alpha > 1$ theorem 2 allows us to obtain multiple interpolants by choosing different β. For any β that divides α, $I_\alpha \Rightarrow I_\beta$ and $I_\beta \Rightarrow I_1$.

4 System of Linear Modular Equations (LMEs)

In this section we discuss proofs of unsatisfiability and interpolation algorithm for LMEs. We first consider a system of LMEs where all equations have the same modulus l, where l is a rational number. We denote this system as $CX \equiv_l D$, where C denotes an $m \times n$ rational matrix, X denotes a column vector of n integer variables and D denotes a column vector of m rational numbers. The next theorem gives a necessary and sufficient condition for $CX \equiv_l D$ to have an integral solution.

Theorem 3. *The system $CX \equiv_l D$ has no integral solution for X if and only if there exists a rational row vector R such that RC is integral, lR is integral, and RD is not an integer.* (The proof uses reduction to LDEs and is given in [10].)

Definition 3. *We say a system of LMEs $CX \equiv_l D$ is* **unsatisfiable** *if it has no integral solution X. A* **proof of unsatisfiability** *for a system of LMEs $CX \equiv_l D$ is a rational row vector R such that RC is integral, lR is integral, and RD is not an integer.*

Example 8. Consider the system of LMEs $CX \equiv_8 D$ and a proof of unsatisfiability R:

$$CX \equiv_8 D := \begin{bmatrix} 2 & 2 \\ 2 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \equiv_8 \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \qquad \begin{array}{ll} R & = [\frac{1}{4}, -\frac{1}{2}, -\frac{1}{8}] \\ RC & = [-1, 0] \\ lR & = [2, -4, -1] \\ RD & = -\frac{3}{2} \end{array}$$

Intuitively, $CX \equiv_8 D$ is unsatisfiable because we can take an integer linear combination of the given equations using lR to get a contradiction $0 \equiv_8 -12$.

Definition 4. *(***Implication***) A system of LMEs $CX \equiv_l D$* **implies** *a LME $AX \equiv_l B$, if every integral vector X satisfying $CX \equiv_l D$ also satisfies $AX \equiv_l B$.*

Lemma 4. *For every* **integral** *row vector U the system of LMEs $CX \equiv_l D$ imply $UCX \equiv_l UD$.*

4.1 Computing Interpolants for Systems of LMEs

Let $AX \equiv_l A'$ and $BX \equiv_l B'$ be two systems of LMEs such that $AX \equiv_l A' \wedge BX \equiv_l B'$ is unsatisfiable. We show that $(AX \equiv_l A', BX \equiv_l B')$ always has a LME as an interpolant. Let $R = [R_1, R_2]$ denote a proof of unsatisfiability for the system $AX \equiv_l A' \wedge BX \equiv_l B'$ such that $R_1A + R_2B$ is integral, $lR = [lR_1, lR_2]$ is integral, and $R_1A' + R_2B'$ is not an integer. The following theorem shows that we can take integer linear combinations of equations in $AX \equiv_l A'$ to obtain interpolants.

Theorem 4. *We assume $l \neq 0$. Let S_1 denote the set of non-zero coefficients of $x_i \in V_{A \setminus B}$ in R_1AX. Let S_2 denote the set of non-zero elements of row vector lR_1. If $S_2 = \emptyset$, then the interpolant for $(AX \equiv_l A', BX \equiv_l B')$ is a trivial LME $0 \equiv_l 0$. Otherwise, let $S_2 \neq \emptyset$. Let α denote the gcd of numbers in $S_1 \cup S_2$. (a) α is an integer and $\alpha > 0$. (b) Let β be any integer that divides α. Let $U = \frac{l}{\beta}R_1$. Then $UAX \equiv_l UA'$ is an interpolant for $(AX \equiv_l A', BX \equiv_l B')$.* (The proof is given in [10].)

Example 9. Consider the system of LMEs $CX \equiv_l D$ in Example 8. Let $AX \equiv_l A'$ denote the first two equations in $CX \equiv_l D$ and $BX \equiv_l B'$ denote the last equation in $CX \equiv_l D$. Observe that $V_{A\setminus B} := \{y\}, V_{AB} := \{x\}, V_{B\setminus A} := \emptyset$. A proof of unsatisfiability for $CX \equiv_l D$ is $R = [\frac{1}{4}, -\frac{1}{2}, -\frac{1}{8}]$. We have $R_1 = [\frac{1}{4}, -\frac{1}{2}]$, $lR_1 = [2, -4]$, R_1AX is $-\frac{1}{2}x$, $S_1 = \emptyset$, $S_2 = \{2, -4\}$, $\alpha = 2$. We can take $\beta = 1$ or $\beta = 2$ to obtain two valid interpolants. For $\beta = 1$, $U = [2, -4]$ and the interpolant $UAX \equiv_l UA'$ is $-4x \equiv_8 -8$ (equivalently $x \equiv_2 0$). For $\beta = 2$, $U = [1, -2]$ and the interpolant $UAX \equiv_l UA'$ is $-2x \equiv_8 -4$ (equivalently $x \equiv_4 2$).

4.2 Handling LMEs with Different Moduli

Consider a system F of LMEs, where equations in F can have different moduli. In order to check the satisfiability of F, we obtain another equivalent system of equations F' such that each equation in F' has the same modulus. This is done using a standard trick. Let $m_1, \ldots, m_k$ represent the different moduli occurring in equations in F. Let m denote the least common multiple of $m_1, \ldots, m_k$. We multiply each equation $t \equiv_{m_i} c$ in F by $\frac{m}{m_i}$ to obtain another equation $\frac{m}{m_i}t \equiv_m \frac{m}{m_i}c$. Let F' represent the set of new equations. All equations in F' have same modulus m. Using basic modular arithmetic one can show that F and F' are equivalent. Suppose F is unsatisfiable. Then the interpolants for any partition of F can be computed by working with F' and using the techniques described in the previous section. For example, let F represent the following system of LMEs $x \equiv_2 1 \wedge x + y \equiv_4 2 \wedge 2x + y \equiv_8 4$. One can work with $F' := 4x \equiv_8 4 \wedge 2x + 2y \equiv_8 4 \wedge 2x + y \equiv_8 4$ instead of F.

5 Algorithms for Obtaining Proofs of Unsatisfiability

Polynomial time algorithms are known for determining if a system of LDEs $CX = D$ has an integral solution or not [20]. We review one such algorithm that is based on the computation of the *Hermite Normal Form (HNF)* of the matrix C.

Using standard Gaussian elimination it can be determined if $CX = D$ has a rational solution or not. If $CX = D$ has no rational solution, then it cannot have any integral solution. In the discussion below we assume that $CX = D$ has a rational solution. Without loss of generality we assume that the matrix C has *full row rank*, that is, all rows of C are linearly independent (linearly dependent equations can be removed).

The HNF of an $m \times n$ matrix C with full row rank is of the form $[E \;\; 0]$ where 0 represents an $m \times (n - m)$ matrix filled with zeros and E is a square $m \times m$ matrix with the following properties: 1) E is lower triangular 2) E is non-singular (invertible) 3) all entries in E are non-negative and the maximum entry in each row lies on the diagonal. The HNF of a matrix can be obtained by three elementary column operations. 1) Exchanging two columns. 2) Multiplying a column by -1. 3) Adding an integral multiple of one column to another column. Each column operation can be represented by a unimodular matrix. A *unimodular matrix* is a square matrix with integer entries and determinant +1 or -1. The product of unimodular matrices is a unimodular matrix. The inverse of a unimodular matrix is a unimodular matrix. The conversion of C to HNF can be represented as follows $CU = [E \;\; 0]$, where U is a unimodular matrix, the sizes of C, U, E are $m \times n, n \times n, m \times m$, respectively and 0 represents an $m \times (n - m)$

matrix filled with zeros ($n \geq m$ because C has full row-rank). The following result shows the use of HNF in determining the satisfiability of a system of LDEs.

Lemma 5. *(Corollary 5.3(b) in [20]) For C, X, D, E defined as above, $CX = D$ has no integral solution if and only if $E^{-1}D$ is not integral. (E^{-1} denotes the inverse of E.)*

Example 10. For the system of LDEs $CX = D$ in example 3 we have the following:

$$\underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}}_{[E \ \ 0]} \qquad \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}}_{E^{-1}} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{D} = \underbrace{\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}}_{not\ integral}$$

5.1 Obtaining a Proof of Unsatisfiability for a System of LDEs

If a system of LDEs $CX = D$ is unsatisfiable, then we want to compute a row vector R such that RC is integral and RD is not an integer. The following corollary shows that the proof of unsatisfiability can be obtained by using the HNF of C.

Corollary 1. *Given $CX = D$ where C, D are rational matrices, and C has full row rank. Let $[E \ \ 0]$ denote the HNF of C. If $CX = D$ has no integral solution, then $E^{-1}D$ is not integral. Suppose the i^{th} entry in $E^{-1}D$ is not an integer. Let R' denote the i^{th} row in E^{-1}. Then (a) $R'D$ is not an integer and (b) $R'C$ is integral. Thus, R' serves as the required proof of unsatisfiability of $CX = D$.*

In example 10 the second row in $E^{-1}D$ is not an integer. Thus, the proof of unsatisfiability of $CX = D$ is the second row in E^{-1} which is $[-\frac{1}{2}, \frac{1}{2}]$.

Proofs of Unsatisfiability for LMEs: Let $CX \equiv_l D$ be a system of LMEs. Each equation $t_i \equiv_l d_i$ in $CX \equiv_l D$ can be written as an equi-satisfiable LDE, $t_i + lv_i = d_i$, where v_i is a new integer variable. In this way we can reduce $CX \equiv_l D$ to an equi-satisfiable system of LDEs $C'Z = D$. The proof of unsatisfiability of $C'Z = D$ is exactly a proof of unsatisfiability of $CX \equiv_l D$ (see the proof of theorem 3 in [10]).

If a system of LDEs or LMEs is unsatisfiable, then we can obtain a proof of unsatisfiability in polynomial time. This is because HNF computation, matrix inversion, and matrix multiplication can be done in polynomial time in the size of input [20].

6 Handling Linear Diophantine Equations and Disequations

We show how to compute interpolants in presence of linear diophantine disequations. A *linear diophantine disequation (LDD)* is of the form $c_1x_1 + \ldots + c_nx_n \neq c_0$, where $c_0, \ldots, c_n$ are rational numbers and $x_1, \ldots, x_n$ are integer variables. *A system of LDEs+LDDs* denotes conjunctions of LDEs and LDDs. For example, $x + 2y = 1 \wedge x + y \neq 1 \wedge 2y + z \neq 1$ with x, y, z as integer variables represents a system of LDEs+LDDs. We represent a conjunction of m LDDs as $\bigwedge_{i=1}^{m} C_iX \neq D_i$, where C_i is a rational row vector and D_i is a rational number. The next theorem gives a necessary and sufficient condition for a system of LDEs+LDDs to have an integral solution.

Theorem 5. *Let F denote $AX = B \wedge \bigwedge_{i=1}^{m} C_i X \neq D_i$. The following are equivalent:*
1. F has no integral solution
2. F has no rational solution or $AX = B$ has no integral solution.

The proof of $(2) \Rightarrow (1)$ in Theorem 5 is easy. The proof of $(1) \Rightarrow (2)$ is involved and relies on the following lemmas (see full proof in [10]). The first lemma shows that if a system of LDEs $AX = B$ has an integral solution, then every LDE that is implied by $AX = B$, can be obtained by a linear combination of equations in $AX = B$.

Lemma 6. *A system of LDEs $AX = B$ implies a LDE $EX = F$ if and only if $AX = B$ is unsatisfiable or there exists a rational vector R such that $E = RA$ and $F = RB$.*

We use the properties of the *cutting-plane* proof system [20, 5] in order to prove lemma 6. The next lemma shows that if a system of LDEs implies a disjunction of LDEs, then it implies one of the LDEs in the disjunction (also called *convexity* [17]).

Lemma 7. *A system of LDEs $AX = B$ implies $\bigvee_{i=1}^{m} C_i X = D_i$ if and only if there exists $1 \leq k \leq m$ such that $AX = B$ implies $C_k X = D_k$.*

We use a theorem from [20] that gives a parametric description of the integral solutions to $AX = B$ in order to prove lemma 7. Let F denote $AX = B \wedge \bigwedge_{i=1}^{m} C_i X \neq D_i$. Using Theorem 5 we can determine whether F has an integral solution in polynomial time. This is because checking if $AX = B$ has an integral solution can be done in polynomial time [20, 5]. Checking whether the system F has a rational solution can be done in polynomial time as well [17].

6.1 Interpolants for LDEs+LDDs

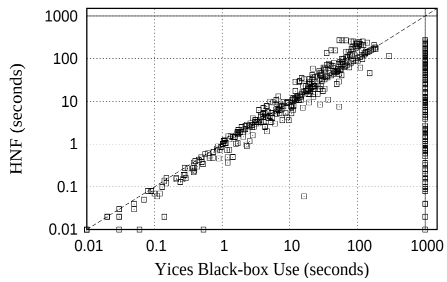
We say a system of LDEs+LDDs is **unsatisfiable** if it has no integral solution. Consider systems of LDEs+LDDs $F := F_1 \wedge F_2$ and $G := G_1 \wedge G_2$, where F_1, G_1 are systems of LDEs and F_2, G_2 are systems of LDDs. $F \wedge G$ represents another system of LDEs+LDDs. Suppose $F \wedge G$ is unsatisfiable. The interpolant for (F, G) can be computed by considering two cases (due to theorem 5):
Case 1: $F \wedge G$ is unsatisfiable because $F_1 \wedge F_2 \wedge G_1 \wedge G_2$ has no rational solution. We can compute an interpolant for (F, G) using the techniques described in [15, 19, 6]. The algorithms in [15, 19, 6] can result in interpolants containing inequalities. We describe an alternative algorithm in [10] that always produces a LDE or a LDD as an interpolant.
Case 2: $F \wedge G$ is unsatisfiable because $F_1 \wedge G_1$ has no integral solution. In this case we can compute an interpolant for the pair (F_1, G_1) using the techniques from Section 3. The computed interpolant will be an interpolant for (F, G). It can be a LDE or a LME.

7 Experimental Results

We implemented the interpolation algorithms in a tool called `INTeger INTerpolate (INT2)`. The experiments are performed on a 1.86 GHz Intel Xeon (R) machine with 4 GB of memory running Linux. `INT2` is designed for computing interpolants for formulas (LDEs, LMEs, LDEs+LDDs) that are satisfiable over rationals but unsatisfiable over integers. Currently, there are no other interpolation tools for such formulas.
Use of Interpolants in Verification: We wrote a collection of small C programs each containing a `while` loop and an ERROR label. These programs are safe (ERROR

Example	Preds/Interpolants	VINT2
ex1	$y \equiv_2 1$	2.72s
ex2	$x + y \equiv_2 0$	0.83s
ex4	$x + y + z \equiv_4 0$	0.95s
ex5	$x \equiv_4 0, y \equiv_4 0$	1.1s
ex6	$4x + 2y + z \equiv_8 0$	0.93s
ex7	$4x - 2y + z \equiv_{2^{22}} 0$	0.54s
forb1	$x + y \equiv_3 0$	-

(a)



(b)

Fig. 1. (a) Table showing the predicates needed and time taken in seconds. (b) Comparing Hermite Normal Form based algorithm and black-box use of Yices for getting proofs of unsatisfiability

is unreachable). The existing tools based on predicate abstraction and counterexample guided abstraction refinement (CEGAR) such as `BLAST` [9], `SATABS` [1] are not able to verify these programs. This is because the inductive invariant required for the proof contains LMEs as predicates, shown in the "Preds/Interpolants" column of Figure 1(a). These predicates cannot be discovered by the interpolation engine [15, 19] used in `BLAST` or by the weakest precondition based procedure used in `SATABS`. The interpolation algorithms described in this paper are able to find the right predicates by computing the interpolants for spurious program traces. Only one unwinding of the `while` loop suffices to find the right predicates in 6 out of 7 cases.

We wrote similar programs in Verilog and tried verifying them with `VCEGAR` [2], a CEGAR based model checker for Verilog. `VCEGAR` fails on these examples due to its use of weakest preconditions. Next, we externally provided the interpolants (predicates) found by `INT2` to `VCEGAR`. With the help of these predicates `VCEGAR` is able to show the unreachability of ERROR labels in all examples except forb1 (ERROR is reachable in the Verilog version of forb1). The runtimes are shown in "VINT2" column.

Müller-Olm and Seidl [16] propose an abstraction technique that can infer linear invariants that are sound with respect to integer arithmetic modulo a power of 2. Their work provides an alternative way of verifying the programs listed in Figure 1(a).

Proofs of Unsatisfiability (PoU) Algorithms: We obtained 459 unsatisfiable formulas (system of LDEs) by unwinding the `while` loops for C programs mentioned above. The number of LDEs in these formulas range from 3 to 1500 with 2 to 4 variables per equation. There are two options for obtaining PoU in INT2. a) Using Hermite Normal Form (HNF) (Section 5.1). We use PARI/GP [4] to compute HNF of matrices. b) By using a state-of-the-art SMT solver Yices 1.0.11 [3] in a black-box fashion (along the lines of [19]). Given a system of LDEs $AX = B$ we encode the constraints that RA is integral and RB is not an integer by means of mixed integer linear arithmetic constraints (see [10]). The SMT solver returns concrete values to elements in R if $AX = B$ is unsatisfiable. The comparison between (a) and (b) is shown in Figure 1(b). There is a timeout of 1000 seconds per problem. The HNF based algorithm is able to solve all problems, while the black-box usage of Yices cannot solve 102 problems within the timeout. Thus, the HNF based method is superior over the black-box use of Yices.

Note that the interpolation algorithms proposed in our paper are independent of the algorithm used to generate the PoU. Any decision procedure that can produce PoU according to definitions 1, 3 can be used (we are not restricted to using HNF or Yices).

8 Conclusion

We presented polynomial time algorithms for computing proofs of unsatisfiability and interpolants for conjunctions of linear diophantine equations, linear modular equations and linear diophantine disequations. These interpolation algorithms are useful for discovering modular/divisibility predicates from spurious counterexamples in a counterexample guided abstraction refinement framework. In future, we plan to work on interpolating theorem provers for integer linear arithmetic and bit-vector arithmetic and make use of the satisfiability modulo theories framework.

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