\( \epsilon \)-descriptiveness of Lipshitz functions on a closed interval

Sa’ar Zehavi

July 9, 2017

Abstract

We define a notion of \( \epsilon \)-approximation and show that the space of Lipshitz functions from \([0,1]\) to \([0,1]\) can be \( \epsilon \)-approximated using just one function, \( G \).

1

Assume a world \( I \) of cardinality at most the continuum. \( I \) can be thought about as a set of chess players (not necessarily countable), such that for every two elements \( i, j \in I \) there exists some \( p_{i,j} \in [0,1] \) which describes the odds of \( i \) defeating \( j \).

Definition 1 Given a function \( F : \mathbb{R} \rightarrow [0,1] \), not necessarily Lipshitz, and merits \( \mu : I \rightarrow \mathbb{R} \), we will say that \( F, \mu \) describes \( I, p \) if \( \forall i, j \in I, F(\mu(i) - \mu(j)) = p_{i,j} \).

Definition 2 Given a world, \( I, p \), we will say that \( G \) \( \epsilon \)-approximates \( F, \mu \), if \( \forall i, j \in I, |F(\mu(i) - \mu(j)) - G(\nu(i) - \nu(j))| \leq \epsilon \).

In the following sections we will be interested in \( \epsilon \)-approximating the set of Lipshitz function on a bounded interval.

Definition 3 Let \( L_C \) be the following function space, \( L_C := \{ f : [0,1] \rightarrow [0,1] \mid \forall x, y \in [0,1] : |f(x) - f(y)| < C|x - y| \} \), namely the \( \text{C-Lipshitz functions} \) over the \([0,1]\) interval.

We are ready to state the main theorem, assuming the existence of a function \( G \) as described in the abstract, which we shall define after stating the theorem.

Theorem 1 There exists a function \( G \), such that for any world \( I \) of at most continuum cardinality, and probabilities \( p \), which are described by a Lipshitz function \( F \in L_C \), with merits: \( \mu \) which satisfy: \( \mu(R) \subseteq [0,1] \). \( G \) \( \epsilon \)-approximates \( F, \mu \).
In order to prove the previous claim we will have to first define $G$, in order to do so, we will need to state and prove some key lemmas.

**Lemma 1** $L_C$ has a finite covering with $\epsilon$-radius balls, with respect to the $L_\infty$ norm.

**Proof 1** It is well known that $L_C$ is compact with respect to the $L_\infty$ norm, via the Arzelà-Ascoli theorem. Let $B(f, \epsilon) = \{g \in L_C||f - g||_\infty < \epsilon\}$, and consider the open covering of $L_C$ by balls of radius $\epsilon$ over all elements $f \in L_C$, i.e. $\bigcup_{f \in L_C} B(f, \epsilon)$. This is an infinite covering of a compact space by open sets, and hence, has a finite covering by the Heine–Borel theorem. I.e., there exists a number $N$ and a family $(f_i)_{i=1}^N$, such that $\forall i \in [N] : f_i \in L_C$. Then it holds that $\bigcup_{i=1}^N B(f_i, \epsilon) = L_C$.

Given the finite covering of $L_C$, with $\epsilon$-radius balls, centered at $(f_i)_{i=1}^N$, it holds that for any $g \in L_C$, $\exists i_0 \in [N]$, such that $||f - g||_\infty < \epsilon$. We will name these $f_i$, the representatives of $L_C$.

**Lemma 2** There exists an algebraically independent set $A \subseteq [0, 1]$ with cardinality of the real line.

Let $A$ be such a set, $|A| = \aleph_0$ implies that $|A^N| = |A|$, and hence $\exists \Phi : [N] \times R \rightarrow A$, such that $\Phi$ is bijective.

**Claim 1** $\forall i, j \in [N], i \neq j$, $\Phi(i, R) \cap \Phi(j, R) = \emptyset$.

**Definition 4** A set $S$ with the operation $(\cdot)$ is said to have unique differences if $\forall a, b, c, d \in S$ such that $a \neq b, c \neq d, a - b \neq c - d$. I.e., the differences of different elements of $S$ are different.

**Claim 2** $\forall i \in [N], \Phi(i, R)$ has unique differences, moreover, $\forall i, j \in [N]$ it holds that $\forall$ distinct $a_i, b_i \in \Phi(i, R), a_j, b_j \in \Phi(j, R) : a_i - b_i \neq a_j - b_j$.

**Proof 2** This holds due to the algebraic independence of each $\Phi(i, R)$, and due to the algebraic independence of $A$.

**Definition 5** Denote the difference set of a set $S$ as $\Delta S$, where $\Delta S := \{a - b | a, b \in S\}$.

Let $i \in [N]$, for each $f_i \in L_C$, define $\tilde{f}_i : \Delta \Phi(i, R) \rightarrow [0, 1]$ in the following way: Given $a - b \in \Delta \Phi(i, R)$, such that $a, b \in \Phi(i, R)$, and assume $(i, x) = \Phi^{-1}(a)$, then, for convenience, denote $x = \Phi^{-1}(a)_2$, we define $\tilde{f}_i(a - b) = f_i(\Phi^{-1}(a)_2 - \Phi^{-1}(b)_2)$.

We are finally ready to define $G$:

**Definition 6** $\forall i \in [N] : G_{|\Phi_i(i, R)} = \tilde{f}_i$.

**Theorem 2** $G$, $\epsilon$-approximates any function $F \in L_C$ with merits $\mu$ that satisfy $\mu(R) \subseteq [0, 1]$. 

2
Proof 3 Given a world $I$, a function $F \in \mathcal{L}$, and merits $\mu$ that satisfy that $\forall \tau \in I : \mu(\tau) \in [0,1]$, we will show that there exists a $\nu : I \rightarrow [0,1]$, such that $\forall \sigma, \tau \in I : |F(\mu(\tau) - \mu(\sigma)) - G(\nu(\tau) - \nu(\sigma))| < \epsilon$. As we know, there exists a $t \in [N]$, such that $\|f_t - F\|_{\infty} < \epsilon$. Let $\tau, \sigma \in I$, set $\nu(\tau) = \Phi(t, \mu(\tau))$, then $|F(\mu(\tau) - \mu(\sigma)) - G(\nu(\tau) - \nu(\sigma))| = |F(\mu(\tau) - \mu(\sigma)) - G(\Phi(t, \mu(\tau)) - \Phi(t, \mu(\sigma)))| = |F(\mu(\tau) - \mu(\sigma)) - \tilde{f}_t(\Phi(t, \mu(\tau)) - \Phi(t, \mu(\sigma)))| = |F(\mu(\tau) - \mu(\sigma)) - \tilde{f}_t(\Phi^{-1}(\Phi(t, \mu(\tau))) - \Phi^{-1}(\Phi(t, \mu(\sigma))))| = |F(\mu(\tau) - \mu(\sigma)) - \tilde{f}_t(\mu(\tau) - \mu(\sigma))| < \epsilon$.

Which establishes the proof, thus $G$ $\epsilon$-approximates any function $F \in \mathcal{L}$ with merits $\mu$ that satisfy $\mu(\mathcal{R}) \subseteq [0,1]$. 
