Precise Continuous Contact Motion Analysis for Freeform Geometry

Yong-Joon Kim
Department of Computer Science
Technion, Israel
Two Main Parts

- Precise contact motion - planar curves.
  Joint Work with Prof. Gershon Elber and Prof. Myung-Soo Kim.

- Precise contact motion - 5-axis machining.
  Joint Work with Prof. Gershon Elber, Michael Barton and Prof. Helmut Pottmann.
Precise Continuous Contact Motion for $C^1$ Planar Curves

Contact is tangential unless otherwise stated.
Motivation

http://en.wikipedia.org/wiki/Gear

http://en.wikipedia.org/wiki/Cam

Center for Graphics and Geometric Computing, Technion
C-Space Obstacle of Planar Curves

1-, 2-, and 3-contacts.
2-Contact Motion

About 4 minutes, Intel i7, single thread.
3-Contact Configuration
Algebraic Condition for 1-Contact

- Assume that $D(v)$ is fixed and $C(u)$ has three degrees of freedom: $x$, $y$ translation and $\theta$ for rotation.

\[
R_\theta [C(u_1)]_x + x = D_x(v_1),
\]
\[
R_\theta [C(u_1)]_y + y = D_y(v_1),
\]
\[
R_\theta [C'(u_1)] \parallel D'(v_1).
\]

- For 1-contact (bivariate solution):

3 Eqns in 5 Unknowns: $u_1$, $v_1$ and $x$, $y$, $\theta$. 
Algebraic Condition for 2-Contact

\[
\begin{align*}
R_\theta \left[ C(u_1) \right]_x + x &= D_x(v_1), \\
R_\theta \left[ C(u_1) \right]_y + y &= D_y(v_1), \\
R_\theta \left[ C'(u_1) \right] \parallel D'(v_1), \\
R_\theta \left[ C(u_2) \right]_x + x &= D_x(v_2), \\
R_\theta \left[ C(u_2) \right]_y + y &= D_y(v_2), \\
R_\theta \left[ C'(u_2) \right] \parallel D'(v_2).
\end{align*}
\]

For 2-contact (univariate solution):

6 Eqns in 7 Unknowns: \( u_1, v_1, u_2, v_2 \) and \( x, y, \theta \).
Algebraic Condition for $K$-Contact

- Assume that $D(v)$ is fixed and $C(u)$ has three degrees of freedom: $x, y$ translation and $\theta$ for rotation.

- The algebraic conditions for a $K$-contact at locations $(u_i, v_i), i = 1, \ldots, K$ are:

  \[
  R_{\theta}[C(u_i)]_x + x = D_x(v_i), \\
  R_{\theta}[C(u_i)]_y + y = D_y(v_i), \\
  R_{\theta}[C'(u_i)] \parallel D'(v_i).
  \]

- For $K$-contact: $3k$ Eqns in $2k + 3$ Unknowns: $u_i, v_i$ and $x, y, \theta$. 
Algebraic Constraints Subdivision Solver

- **Input:** A set of tensor product multivariate Bezier/B-spline form equations.
  
  \[ F_1(x_1, x_2, ..., x_n) = ... = F_m(x_1, ..., x_n) = 0. \]

- **Subdivision step:**
  
  - Purge domains in which no solution can exist
    (i.e. no sign change in coefficients – convex hull property)
  
  - Termination condition (I.e. topological guarantee, etc.)
    
    - Go to the numeric improvement step if it satisfy the condition.
  
  - Subdivide the domain otherwise.
Subdivision Termination Condition 1

Lemma 1: Given a convex curve segment $C(u)$ and a concave segment $D(v)$, if there exists a 2-contact configuration for $C(u)$ and $D(v)$, there exist at least two different solutions for the following two equations.

\[ k_C(u) - k_D(v) = 0, \]
\[ C'(u) \parallel D'(v). \]
Subdivision Termination Condition 2

- For a given pair of curves $C(u)$ and $D(v)$, we seek a subdivision of $C(u)$ and $D(v)$ until the two contact points are in a separated curve.
- Using Lemma 1, we stop when lemma no longer holds.
Domain Purging 1

- We seek to identify regions in the domain that can not contribute to the C-space obstacle boundary.
- The solutions in the interior of the C-space obstacle boundary cause an inter-penetration of $C(u)$ into $D(v)$. 
Domain Purging 2

For a given sub-domain $D_s = [u, \bar{u}] \times [v, \bar{v}] \times [\theta, \bar{\theta}]$
we would like to know if $D_s$ can be purged completely.

\[ R_{\theta} \left[ C(u) \right]_x + x = D_x (v), \]
\[ R_{\theta} \left[ C(u) \right]_y + y = D_y (v), \]
\[ R_{\theta} \left[ C'(u) \right] \parallel D'(v). \]

We define the set of transformation, over $D_s$: \[
\pi_{D_s} = \{ T(x, y, \theta) \vert x = x(u, v, \theta), \\
y = y(u, v, \theta), (u, v, \theta) \in D_s \}. \]
Domain Purging 3

- Let the maximum penetration depth of $C(u)$ into $D(v)$ for a transformation $T(x, y, \theta) \in \pi_{DS}$ be $PD(T(x, y, \theta))$.
- $PD(T(x, y, \theta))$ is positive if $C(u)$ inter-penetrates $D(v)$ and negative if $C(u)$ and $D(v)$ are separated.
- Let $PD(\pi_{DS})$ be the minimum $PD(T(x, y, \theta))$ over the entire set of transformations $\pi_{DS}$,

$$PD(\pi_{DS}) = \min\{PD(T(x, y, \theta)) \mid T(x, y, \theta) \in \pi_{DS}\}.$$
Domain Purging 4

- If $PD(\pi_{DS})$ is positive, all the solutions from the subdomain $D_S$ have an inter-penetration and thus invalid. 
  Redundant domain that can be purged!

- Computing $PD(\pi_{DS})$ is computationally expensive.
- If lower bound of $PD(\pi_{DS})$ is positive, we can purge the domain. (Details for computing the lower bound is in the paper.)
Domain Purging 5

Even after the domain purging, we still have some redundant parts in the solutions.
3-Contact Points

Given the collision free 2-Contact motions (with some redundancy), we derive the precise 3-Contact points by intersecting the neighborhoods of the redundancies.
Special Events 1
A Third Order Contact (Curvature Derivative Match)
Special Events 2

- A Third Order Contact (Curvature Derivative Match)

\[ k_C(u) - k_D(v) = 0 \]
\[ k'_C(u) - k'_D(v) \frac{\partial v}{\partial u} = 0 \]

- Position and tangent direction can be always matched by translating and rotating \( C(u) \) to \( D(v) \).
Results 1

About 6 minutes, Intel i7, single thread.
Results 2

About 20 minutes, Intel i7, single thread.
Motion Planning

One can use graph search algorithms (i.e. Dijkstra), on the C-space graph, to seek optimal motion.

About 25 minutes, Intel i7, single thread.
More Complex Motion Planning

About one hour, Intel i7, single thread.
Tool Path Contact Computation for 5-Axis CNC Machining

- Two degrees of freedom to orient the tool.
- Assumes a cylinder tool.
  - Bull-nose tool reduced to a cylinder via surface offset.
- Seeks best matching between tool and $C^1$ surface.
Motivation

- State of the art:
  Curvature Matched Machining
  - Jensen, 1993
  - Yoon et al, 2003

- Machining with Hyper-Osculating Circle
  - Match also curvature derivative (zero).
  - Wang et al, 1993
  - Barton et al, 2013
Consider a first contact location, $P_c$.

The tool is fixed at $P_c$.

Consider tool tilt $\phi$ & rotate $\theta$ (two DOFs) around $P_c$. 
Contact Analysis II

**Gouging Region**

Curvature matched

2-Contact

Collision -Free Region

2-contact gouging point
Curvature matched point
2-contact gouging-free
Hyper-osculating contact
Consider a tool bottom circle $C_t$ with radius $r_T$. Let $C_s$ be the intersection curve between the target surface and the plane containing $C_t$. 
Hyper Osculating Circles II

HOCs satisfy two conditions.

- $C_s$ and $C_t$ share the same curvature at $P_c$.
  - From Meusnier’s theorem,
    \[
    \frac{1}{(k_1 \cos^2 \theta + k_2 \sin^2 \theta)} \cos \phi = r_T,
    \]
    where $k_1$ and $k_2$ are principle curvatures at $P_c$.

- $C_s$ and $C_t$ share the same curvature derivative at $P_c$.
  - In other word, $P_c$ is a curvature extreme point of $C_s$.
    - Third order analysis is required to formulate this condition (details are in the paper).
2-contact Configurations I
2-contact Configurations II

Parameterize $P_2$ by $S(u_2, v_2)$, and let

- $r_T$ be the tool radius,
- $N(u_2, v_2)$ be the surface normal,
- $Z(\theta, \phi)$ and $M_T(\theta, \phi)$ be the axis and bottom center of the tool.

Then, this 2-contact case can be formulated algebraically as:

$$\| S(u_2, v_2) - M_T(\theta, \phi) \|^2 - r_T^2 = 0,$$

$$\langle (S(u_2, v_2) - P_C), Z(\theta, \phi) \rangle = 0,$$

$$\langle (S(u_2, v_2) - M_T(\theta, \phi)) \times Z(\theta, \phi), N(u_2, v_2) \rangle = 0,$$

with $u_2$, $v_2$, $\theta$, $\phi$ as unknowns.
Collision Detection / Gouging I

- Tool is always in contact with the target surface, at the contact (milling) location $P_C$.

- Hence, gouging is very difficult to detect by conventional collision detection algorithms.

- Instead, we use maximum penetration depth analysis.
Collision Detection / Gouging II
Collision Detection / Gouging III

Consider the coordinate system of $T$ and let $S^c(u,v)$ and $N^c(u,v)$ be the $S$ and $N$ in the canonical space. Then this maximum penetration depth condition can be formulated as follows:

$$S^c_x(u,v)^2 + S^c_y(u,v)^2 - (r^c_T - S^c_z(u,v))^2 = 0,$$
$$N^c_y(u,v)S^c_x(u,v) - N^c_x(u,v)S^c_y(u,v) = 0,$$

with $u$, $v$ as unknowns.
Tool Path Construction

- A Dijkstra’s algorithm is applied on a bivariate solution set, as a function of $t$ and $\theta$, computing $\varphi$.
- The weight of each edge reflects on the:
  1. Curvature matching quality (difference between tool radius and surface’s radius of curvature).
  2. Ensuring gouging free tool-path & limited angular change.
- Interleaving between hyper-osculation and 2-contacts.
Experimental Results

Blue – hyper-osculation; Yellow – 2-contact.
Experimental Results

Full red color represents an over-cut by 1% of maximum length of the bounding box.

2-Contact/Hyper-Osculation  Curvature matched machining
Using Moduleworks 5-axis CNC simulator.
Experimental Results

10 minutes per path x 99 paths, on Intel i7 (single core)
Simulated using Moduleworks 5-axis CNC simulator.
Conclusions I

2-contact precise motion for planar $C^1$ curves:
- Topological Structure of 2-/3-Contact Configuration
- Precise Solution from Algebraic Constraint.
- Efficient computation via domain purging and subdivision termination condition.
- Path Planning using C-Space graph structure of solution.
- Only compute 2- and 3-contact analysis and greedy 1-contacts traces in bivariate solution spaces.
- Extendible to 3-space motion of surfaces.
Conclusion II

2-contact tool path for 5-axis machining:

- Analysis of curvature matching, 2-contacts and hyper-osculating configurations.
- Collision free higher-order/2-contact tool path for $C^1$ surfaces.
- Efficient tool-path via global optimization, interleaving between 2-contacts and hyper-osculation.
- So far only sampled contact point set is validated.
- Expensive computation that better be made more efficient, for practical use.
Thank you

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