Isogeometric Analysis: Cross Research Between CAGD and PDEs based simulations, an example: Schwarz Additive DD

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(Joint work with Ilya Soloveichick)
DD, IGA and Schwarz

- Solid Modelling (CSG)
- Schwarz Additive Domain Decomposition and Isogeometric Analysis

All computations were done using GeoPDES 1.2
DD, IGA and Schwarz

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- Boundary Conditions
- 1D numerical results

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- Can we prove convergence?
- 2D examples and application to local zooming
- 3D heat and elasticity examples
- Parallelisation
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- Parallelisation
- Current Research and Conclusion

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Isogeometric Mapping

\[
\{B_i\}_{i=1, \ldots, N_0}
\]

parametric domain $\hat{\Omega}$

\[
\text{push forward of } \{B_i\}_{i=1, \ldots, N_0}
\]

physical domain $\Omega$
Isogeometric Mapping

parametric domain $\hat{\Omega}$

$\{B_i\}_{i=1,\ldots,N_1}$

physical domain $\Omega$

push forward of $\{B_i\}_{i=1,\ldots,N_1}$
Isogeometric Mapping

parametric domain $\hat{\Omega}$

physical domain $\Omega$

$\{B_i\}_{i=1,...,N_2}$

push forward of $\{B_i\}_{i=1,...,N_2}$
Example of Tensor Product Domain

Control Lattice and Mesh
Solid Modelling

- Constructive Solid Geometry (CSG) relies on boolean operations of primitives

Example of Boolean Union
Solid Modelling

- Constructive Solid Geometry (CSG) relies on boolean operations of primitives.

More Complex Construct: Union and Subtraction
CSG is only one of, but an important one, tool used by designers.

To define the isogeometric mapping for these complex structures one needs the latest results like


Research on DD and IGA is growing, see the conference by L. Beirao da Vega this morning and the present Mini Symposium!
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To define the isogeometric mapping for these complex structures one needs the latest results like

Another alternative:

- Using Domain Decomposition Methods for CSG defined solids.


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Fig. 3. Modeling the Kitten model using manifold T-spline with 765 control points

Yin He and all, Stony Brooks.
WHAT PROPERTIES DO WE DEMAND?
WHAT PROPERTIES DO WE DEMAND?

- Non-matching meshes
- Easy parallelisation
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- Non-matching meshes
- Easy parallelisation

Schwarz Additive Domain Decomposition fits
Consider the equation $\Delta u = 0, u|_\Omega = 0$ on the domain given by the logo of Domain Decomposition Organization, which is the union of a circle and an overlapping rectangle: $\Omega = \Omega_1 \cup \Omega_2$. The boundary of $\Omega$ is $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1$ (resp. $\Gamma_2$) is the boundary of $\Omega_1 \setminus \Omega_2$ (resp. $\Omega_2 \setminus \Omega_1$).
Classic Reminder
Define a bilinear form $a(\cdot, \cdot) : \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega) \to \mathbb{R}$ and a functional $L : \mathcal{H}^1(\Omega) \to \mathbb{R}$ as:

$$a(w, u) = \int_{\Omega} \nabla w \cdot \nabla u \, d\Omega,$$

(1)

and

$$L(w) = \int_{\Omega} wf \, d\Omega + \int_{\Gamma_N} wh \, d\Gamma.$$

(2)

Now the weak form reads as:

$$a(w, u) = L(w),$$

(3)
Since we assume the domain boundaries to be piecewise smooth and the solutions belong to $H^1(\Omega_j)$ spaces, the Trace theorem provides us with the definition of the trace operator:

\[ P_i : H^1(\Omega_j) \to L^2(\Gamma_i) \]

\[ u_j \to u_j|_{\Gamma_i}. \]
There exist an extension operators:

\[ E_i : \mathcal{H}^1(\Gamma_i) \to \mathcal{H}^1(\Omega_i) \]

\[ \nu_i \to u_i \text{ such that } u_i|_{\Gamma_i} = \nu_i, u_i|_{\Gamma_i} = g|_{\Gamma_i}. \]

Here we assume that the conditions \( u_i|_{\Gamma_i} = \nu_i, u_i|_{\Gamma_i} = g|_{\Gamma_i} \) define some continuous \( \mathcal{H}^1(\Omega_i) \) function.

The continuous version of the ASDDM follows.
DD continuous case

Given initial guesses $u_i^0 \in \mathcal{H}^1(\Omega_i), i = 1, 2$, such that
$u_i^0|_{\Gamma_i} = g|_{\Gamma_i}, i, j = 1, 2$

While convergence conditions are not met:

Find $u_i^n \in \mathcal{H}^1(\Omega_i)$ such that $u_i^n|_{\Gamma_i} = g|_{\Gamma_i}$ and

$$a_i(u_i^n - E_i P_i u_j^{n-1}, v_i) = L_i(v_i) - a_i(E_i P_i u_j^{n-1}, v_i)$$

for any $v_i \in \mathcal{H}_0^1(\Omega_i), i, j = 1, 2, i \neq j$. 

ASDDM algorithm
DD and IGA, finite dimensional case

Given $V_i \subset H_0^1(\Omega_i) = [\Phi_{(i,j)}(x) = B_j(F_i^{-1}(x)); ]$, set $u_1^0 = 0, u_2^0 = 0$;

Given $u_i^0 \in V_i, i = 1, 2$, such that $u_i^0|_{\Gamma_{ij}} = g|_{\Gamma_{ij}}, i = 1, 2$, for $i = 1, 2; j = 1, 2, i \neq j$

While convergence conditions are not met:

Find $u_i^n \in V_i$ such that $u_i^n|_{\Gamma_i} = g|_{\Gamma_i}$, and

$$a_i(u_i^n - E_i P_i u_j^{n-1}, v_i) = L_i(v_i) - a_i(E_i P_i u_j^{n-1}, v_i)$$

for any $v_i \in V_i$.

ASDDM algorithm
Figure 1 illustrates the described ASDDM applied to a two-domain problem. The algorithm involves the following steps:

**Iteration $n$**
- Calculate the solution $u_1$ in domain $\Omega_1$.
- Project the solution $u_2$ onto the boundary of $\Omega_1$.
- Approximate the boundary conditions by a NURBS curve.
- Extend the NURBS curve to the whole domain $\Omega_1$.
- New boundary conditions for $\Omega_1$.
- Calculate the solution $u_1$ in domain $\Omega_1$.
- Project the solution $u_2$ onto the boundary of $\Omega_1$.

**Iteration $n+1$**
- Calculate the solution $u_2$ in domain $\Omega_2$.
- Project the solution $u_1$ onto the boundary of $\Omega_2$.
- Approximate the boundary conditions by a NURBS curve.
- Extend the NURBS curve to the whole domain $\Omega_2$.
- New boundary conditions for $\Omega_2$.
- Calculate the solution $u_2$ in domain $\Omega_2$.
- Project the solution $u_1$ onto the boundary of $\Omega_2$. 
We test the convergence on 

\[ \| u_{1,\Omega_1 \cap \Omega_2}^m - u_{1,\Omega_1 \cap \Omega_2}^{m-1} \| \text{ and } \| u_{2,\Omega_1 \cap \Omega_2}^m - u_{2,\Omega_1 \cap \Omega_2}^{m-1} \| \]

There is no notion of a global approximate solution, on the overlap. We define the global solution by choosing the subdomain iterative solution on each subdomain and any weighted average within the overlap:

\[ u^n = \chi_1 v^n + \chi_2 w^n, \]

where \( \chi_1 = 1 \) on \( \Omega_1 \setminus (\Omega_1 \cap \Omega_2) \), \( \chi_2 = 1 \) on \( \Omega_2 \setminus (\Omega_1 \cap \Omega_2) \) and \( \chi_1 + \chi_2 = 1 \) on \( \Omega_1 \cap \Omega_2 \).

This approach extends easily to multi-patched domain decomposition.
Overlapping Schwarz and BC

Construction of the Trace Operator
We tested two approaches.
  . Computation in the parametric space
  . Computation in the physical space
Parametric space approach

Pre-Image computation
Overlapping boundaries are trimming curves in the parametric space.

Given any point \((x, y)\) on the boundary \(\Gamma_j^i\) of \(\Omega_j\) : compute its pre-image coordinates \((\xi, \eta)\) in the parametric space \(\hat{\Omega}_i\).

Evaluate the solution \(u^m_i\) at that given \((\xi, \eta)\) point at each iteration.
Physical space approach

Second approach: work only in the physical space.

The solution $u_i$ in the domain $\Omega_i$ is tabulated at some mesh coordinates $(X, Y)$ (say, integration points). For any point $(x, y)$ at which this solution is to be evaluated (the integration points of the boundary of the sub-domain $\Omega_j$) we interpolate by linear or cubic polynomials.

This does not affect the rate of the iterative convergence significantly.
In both approaches we have a discrete collection of values on the boundary $\Gamma_i^j$.

We still have to convert this information into boundary B-spline degrees of freedom.

We considered two methods:
- Least-square approximation of the Dirichlet BC
- Quasi Interpolation

There are also other approaches to impose the Dirichlet boundary conditions such as local least-squares (Govindjee et All.) or Nitsche’s method (Harari)
Least-squares approximation of the Dirichlet BC

Find such coefficients \( \{ \tilde{q}_i \}_{i=N_h+1}^{N_h+N_b^b} \) that minimize the following integral:

\[
\min_{\{q_i\}_{i=N_h+1}^{N_h+N_b^b}} \int_{\Gamma_D} (g(x) - \sum_{N_h+1}^{N_h+N_b^b} q_i \phi_i(x))^2\,d\Gamma.
\]

In practice we will compute the integral using numerical formula, such as Gaussian quadrature, hence \( x \) will be computed at a predetermined collection of values, in the parametric domain.
Non-homogeneous Dirichlet BC in IGA

Quasi interpolation of the Dirichlet BC

Consider a point-wise approximation of $g(x)$.

Assume that the "intersectting" boundary $\Gamma$ corresponds to one side of the parametric domain with $N$ degrees of freedom.

Given $N$ points on this boundary, solve the system of linear equations:

$$
\sum_{j=1}^{N} q_j \phi_j(x_i, y_i) = g(x_i, y_i), \quad i = 1 \ldots N.
$$

Applying the same algorithm to all the sides of the domain where the Dirichlet BC are imposed, gives the value of the corresponding degrees of freedom.
Non-homogeneous Dirichlet BC in IGA

Quasi interpolation of the Dirichlet BC

Take \( \{(x_i, y_i)\}_{i=1}^{N} \) as the images of the centers of the knot spans \( \Rightarrow \) they are uniformly distributed in the parametric space.

These points may be chosen in different ways (e.g., uniform chord length).

One of the open questions we are interested in is to choose these points optimally, given the geometry.
Dirichlet BC, 1D example

1D Domain Decomposition example:
two overlapping domains, non-matching grid.

\[ \Delta u = -1, \]

\[ u(0) = u(1) = 0. \]
Uniform open knot vectors: $\Xi_1 = [0, \ldots \beta]$ and $\Xi_2 = [\alpha, \ldots 1]$; B-splines of degree $p_1$ and $p_2$ on the subdomains $\Omega_1$ and $\Omega_2$, respectively; $N_{h1}$ and $N_{h2}$, the resp. numbers of degrees of freedom

The mapping $F$ is the identity mapping, (parametric space and physical space are the same)

Let $v$ (resp. $w$) be the solution on $\Omega_1$ (resp. $\Omega_2$). Formally we discretize:

$$v^n_{xx} = f \text{ on } \Omega_1, \quad v^n(0) = 0, \quad v^n(\beta) = w^{n-1}(\beta). \quad (4)$$

$$w^n_{xx} = f \text{ on } \Omega_2, \quad w^n(0) = 0, \quad w^n(\alpha) = v^{n-1}(\alpha). \quad (5)$$
The basis functions are numbered by the order of the knot vectors. Only the first and last basis functions are interpolatory, and all the others vanish at the boundary $t = 0$ and $t = 1$.

Consider subdomain $\Omega_1$. The first (resp. last) basis function satisfies $\phi_1(0) = 1$, (resp. $\phi_{N_{h_1}}(\beta) = 1$.)

The approximation operators $A_i$ are the identities.

Given the solution $\tilde{w}^{n-1} = \sum_{i=1}^{N_{h_2}} \psi_i w_i^{n-1}$ at boundary point $\beta$ we project it on the second subdomain: $v_{N_{h_1}}^n = P_1(\tilde{w}^{n-1}(\beta))$, where $P_1$ is the trace operator.

The same reasoning applies on $\Omega_2$. 
Denote the vector of degrees of freedom of the solution $\tilde{w}$ (resp. $\tilde{v}$) as $w$, (resp. $v$.)

We can now build a matrix representation of the iterations . The degree of B-splines on $\Omega_2$ is $p_2$ thus there are no more than $p_2 + 1$ non-zero basis functions $\psi_i$ at the point $\beta$. We may use different trace operators in order to project the solution on the subdomain $\Omega_2$ onto the boundary $\Gamma_1^2$ of the subdomain $\Omega_1$.

We will consider both cases: the exact trace operator and the interpolation trace operator
For this trivial trace operator, the value $v_{N_{h_1}}^n$ we get is:

$$v_{N_{h_1}}^n = P_1(\tilde{w}^{n-1}(\beta)),$$

where $P_1$ is the trace operator.

The operator matrix for the boundary of the $\Omega_1$ subdomain is:

$$v_{N_{h_1}}^n = P_1(\tilde{w}^{n-1}(\beta)) = P_1^e \cdot \begin{pmatrix}
0 \\
\vdots \\
w_i^{n-1} \\
w_{i+1}^{n-1} \\
\vdots \\
w_i^{n-1} \\
w_{i+p_2}^{n-1} \\
\vdots \\
0
\end{pmatrix} \cdot \begin{pmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
w_i^{n-1} & w_{i+1}^{n-1} & \vdots & \psi_i(\beta) & \psi_{i+1}(\beta) & \cdots & \psi_{i+p_2}(\beta) & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
w_i^{n-1} & w_{i+p_2}^{n-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},$$

where $P_1^e \in \mathbb{R}^{N_{h_1} \times N_{h_2}}$. 
Denote by $A_1$ the stiffness matrix for the first subdomain. We partition the sets of indices of basis functions $\mathcal{J} = \{1, 2, \ldots, N_{hj}\}, j = 1, 2$ into two subsets. The subset $\mathcal{I} \subset \mathcal{J}$ of the inner degrees of freedom and $\mathcal{B} \subset \mathcal{J}$ of the boundary degrees of freedom.

In the one dimensional case $\mathcal{B} = \{1, N_{hj}\}$ and $\mathcal{I} = \mathcal{J} \setminus \{1, N_{hj}\} = \{2, 3, \ldots, N_{hj} - 1\}, j = 1, 2.$

The restriction of the stiffness matrix corresponding to the inner degrees of freedom is $A_1(\mathcal{I}, \mathcal{I})$.

When we impose the Dirichlet boundary conditions on some of the degrees of freedom, we have to subtract the corresponding values from the degrees of freedom of the basis functions $\{\phi\}_{i=N_{h1}-k}$ which support intersects with the support of $\phi_{N_{h1}}$. This corresponds to the thickness of the interface in standard IGA-DD methods ...
Now the discretized equation for the subdomain $\Omega_1$ can be rewritten as:

$$
\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & A_1(\mathcal{I}, \mathcal{I}) & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \psi_i(\beta) & \psi_{i+1}(\beta) & \cdots & \psi_{i+p_2}(\beta) & \cdots & 0
\end{pmatrix} \cdot \mathbf{v}^n = 
\begin{pmatrix}
0 \\
f_1 \\
0 \\
0 & -A_1(\mathcal{I}, \mathcal{B}) & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix} \cdot 
\begin{pmatrix}
0 \\
f_1 \\
0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0
\end{pmatrix} \cdot \mathbf{w}^{n-1},
$$

where the vector $\mathbf{f}_1$ corresponds to the inner degrees of freedom of the first subdomain.
Exactly the same reasoning can be applied to the second subdomain $\Omega_2$ to get the following discretized equation:

$$\begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & A_2(I, I) & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 1
\end{pmatrix} \cdot w^n = \begin{pmatrix}
0 \\
0 \\
A_2(I, B) \\
0
\end{pmatrix} + \begin{pmatrix}
0 & \ldots & 0 \\
1 & \ldots & 0 \\
0 & \ldots & 1
\end{pmatrix} \cdot \begin{pmatrix}
\phi_j(\alpha) & \phi_{j+1}(\alpha) & \ldots & \phi_{j+p_1}(\alpha) & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix} \cdot v^{n-1}.$$  

We regroup all the definitions and steps:

$$P = \begin{pmatrix}
O & 0 & \ldots & 0 \\
0 & \phi_j(\alpha) & \ldots & \phi_{j+p_1}(\alpha) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0
\end{pmatrix},$$

$$P = \begin{pmatrix}
0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 0 & \ldots & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \psi_j(\beta) & \ldots & \psi_{j+p_2}(\beta) & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}.$$  

with $P_1^e \in \mathbb{R}^{N_{h1} \times N_{h2}}$, $P_2^e \in \mathbb{R}^{N_{h2} \times N_{h1}}$. 

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For the stiffness matrices we have:

\[
A = \begin{pmatrix}
1 & \cdots & 0 \\
0 & A_1(I,I) & 0 \\
0 & \cdots & 1 \\
O & 1 & \cdots & 0 \\
O & 0 & \cdots & 1
\end{pmatrix} = \begin{pmatrix}
\tilde{A}_1 & O \\
O & \tilde{A}_2
\end{pmatrix},
\]

with \(\tilde{A}_1 \in \mathbb{R}^{N_{h1} \times N_{h1}}, \tilde{A}_2 \in \mathbb{R}^{N_{h2} \times N_{h2}},\)

\[
f = \begin{pmatrix}
0 \\
f_1 \\
0 \\
f_2
\end{pmatrix},
\]

\[
A_{dir} = \begin{pmatrix}
1 & \cdots & 0 \\
0 & -A_1(I,B) & 0 \\
0 & \cdots & 1 \\
O & 1 & \cdots & 0 \\
O & 0 & \cdots & 1
\end{pmatrix} = \begin{pmatrix}
\tilde{A}_{dir1} & O \\
O & \tilde{A}_{dir2}
\end{pmatrix},
\]

with \(\tilde{A}_{dir1} \in \mathbb{R}^{N_{h1} \times N_{h1}}, \tilde{A}_{dir2} \in \mathbb{R}^{N_{h2} \times N_{h2}}.\)
Let us denote the vector of degrees of freedom of the “whole” approximate solution $\tilde{u}^n$ as $u^n = \begin{pmatrix} v^n \\ w^n \end{pmatrix}$. Now, we can unite the two equations (6) and (7) into a single matrix equation

$$A \cdot u^n = f + A_{\text{dir}} \cdot P \cdot u^{n-1}, \quad (9)$$

The iterative scheme is given by:

$$\begin{pmatrix} \tilde{A}_1 & O \\ O & \tilde{A}_2 \end{pmatrix} \cdot u^n = f + \begin{pmatrix} \tilde{A}_{\text{dir}1} & O \\ O & \tilde{A}_{\text{dir}2} \end{pmatrix} \cdot \begin{pmatrix} O & P^e_1 \\ P^e_2 & O \end{pmatrix} \cdot u^{n-1} =$$

$$f + \begin{pmatrix} O & \tilde{A}_{\text{dir}1} P^e_1 \\ \tilde{A}_{\text{dir}2} P^e_2 & O \end{pmatrix} \cdot u^{n-1}, \quad (10)$$
Here we have a simple linear operator.
The algorithm works as was explained above: consider the first subdomain $\Omega_1 = [0, \beta)$. In order to construct a linear interpolation of the solution $\tilde{w}^{n-1}$ at the point $\eta = \beta$ we need to find the knot span $[\eta_i, \eta_{i+1})$ which contains $\beta$. We take the values of the function $\tilde{w}^{n-1}$ at the ends of this chosen span interval $\tilde{w}^{n-1}(\eta_i)$ and $\tilde{w}^{n-1}(\eta_{i+1})$ and their weighted sum:

$$v^n_{\mathcal{N}_h1} = \frac{\beta - \eta_i}{\eta_{i+1} - \eta_i} \tilde{w}^{n-1}(\eta_i) + \frac{\eta_{i+1} - \beta}{\eta_{i+1} - \eta_i} \tilde{w}^{n-1}(\eta_{i+1}).$$

We see that the linear interpolation trace operator is, actually, a convex sum of the two exact interpolation operators corresponding to the points $\eta_i$ and $\eta_{i+1}$:

$$P_1^j = \frac{\beta - \eta_i}{\eta_{i+1} - \eta_i} P^e(\eta_i) + \frac{\eta_{i+1} - \beta}{\eta_{i+1} - \eta_i} P^e(\eta_{i+1}).$$ (11)

Consequently, the matrix of this interpolation trace operator can be obtained as the convex sum of the matrices of the exact trace operators at the points $\eta_i$ and $\eta_{i+1}$.

$$P_1^j = \frac{\beta - \eta_i}{\eta_{i+1} - \eta_i} \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \psi_i(\eta_i) & \psi_{i+1}(\eta_i) & \cdots & \psi_{i+p_2}(\eta_i) & \cdots & 0 \\ \end{pmatrix}$$

$$+ \frac{\eta_{i+1} - \beta}{\eta_{i+1} - \eta_i} \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \psi_{i+1}(\eta_{i+1}) & \psi_{i+2}(\eta_{i+1}) & \cdots & \psi_{i+p_2+1}(\eta_{i+1}) & \cdots & 0 \\ \end{pmatrix}.$$ (12)
The iterative scheme is now:

$$
\begin{pmatrix}
\tilde{A}_1 & O \\
O & \tilde{A}_2
\end{pmatrix} \cdot \mathbf{u}^n = f + \begin{pmatrix}
\tilde{A}_{dir1} & O \\
O & \tilde{A}_{dir2}
\end{pmatrix} \cdot \begin{pmatrix}
O & P^l_1 \\
P^l_2 & O
\end{pmatrix} \cdot \mathbf{u}^{n-1} = f + \begin{pmatrix}
O & \tilde{A}_{dir1} P^l_1 \\
\tilde{A}_{dir2} P^l_2 & O
\end{pmatrix} \cdot \mathbf{u}^{n-1},
$$

(13)

where

$$
P^l_2 = \frac{\alpha - \xi_j}{\xi_{j+1} - \xi_j} P^e(\xi_j) + \frac{\xi_{j+1} - \alpha}{\xi_{j+1} - \xi_j} P^e(\xi_{j+1}).
$$

(14)

and $\alpha$ belongs to the knot span $[\xi_j, \xi_{j+1})$
Dirichlet BC, 1D example

1D example using the parametric space approach
Convergence of the one-dimensional SADDM with non-zero initial guesses. $\tilde{v}^0(\beta) = \beta$ and $\tilde{w}^0(\alpha) = \alpha$
1D example and Overlapping

Iteration speed vs % of overlapping of domains.

![Graph showing L2 error depending on the overlap area with different overlap percentages.](image-url)
DD iteration and degree of approximation

The impact of the degree $p$ of the B-splines on the iterative convergence.

![Iterative L2 error depending on the degree](image)
Why are all our examples so good?

- Proof of convergence of Schwarz Additive Domain Decomposition on non matching grids relies on the maximum principle.

\[ \Omega_1 \]

\[ 0 \quad \alpha \quad \beta \quad 1 \]

\[ \Omega_2 \]

$V_n(C)$ is the $n$-dimensional vector space of column vectors $x = [x_1, x_2, \ldots, x_n]^T$ with complex components. 

$x = \sum_{i=1}^n x_i e_i$, where $\{e_i\}_{i=1}^n$ is the canonical basis for $V_n(C)$.

Let $S$ denote any $r$-dimensional linear subspace of $V_n(C)$ spanned by $r$ vectors $e_i$, $1 \leq r \leq n$.

Associated with $S$ is the Projection operator $P$:
Its matrix is $P_S = (d_1, d_2, \ldots, d_n)$,
where $d_i = 1$ if $e_i \in S$, and zero otherwise.

$P_S x \in S$ and $||P_S x||_\infty \leq ||x||_\infty$ for any $x \in V_n(C)$, so that $||P_S||_\infty \leq 1$

A matrix $A \in \mathbb{R}^{n \times n}$ is normalized with respect to the subspace $S$ 
($A \in \mathcal{N}_S$) if $A \xi = P_S \xi$, where $\xi = (1, 1, \ldots, 1)$

The row sum $\sum_{j=1}^n a_{ij}$ is, respectively unity , (resp. zero)
if $e_i \in S$ (resp. $\notin S$).

Now by a classic result of Varga ,if $A \in \mathcal{N}_S$, and $A$ is a M-matrix, then $A$ satisfies the Maximum Principle .
We consider $\Omega = [0, 1]$, and a corresponding B-Spline Basis, $B_i(t)$. We will assume that $F$ is the identity.

$$\Delta u = -1; u(0) = u(1) = 0. \quad (*)$$

Let $n$ be the dimension of the discrete space given by the Galerkin method, and let $A$ the resulting $(n \times n)$ stiffness matrix.

We enumerate all the basis functions by following the order of the knot vectors.

Only the first and last basis functions are interpolatory, and all the other basis functions vanish at the boundary $t = 0$ and $t = 1$. The stiffness matrix is:

$$K = \begin{pmatrix}
1 & \ldots & 0 \\
0 & \qquad & \qquad \\
& & \qquad \\
\end{pmatrix}
\begin{pmatrix}
A(\mathcal{I}, :)
\end{pmatrix}
\begin{pmatrix}
0 \\
\ldots \\
1
\end{pmatrix},$$

where $A = \{a_{ij}\}_{i,j=1}^n$ $A(\mathcal{I}, :)$ is the stiffness matrix of the problem and $\mathcal{I}$ denotes the rows corresponding to the inner degrees of freedom.

We replaced the first row by the row $(1, 0, \ldots, 0)$ and the last row by the row $(0, \ldots, 0, 1)$ since we have Dirichlet boundary conditions at these degrees of freedom.

We analyze $A(\mathcal{I}, :)$ and $K$ for B-spline basis functions of order $p = 1, 2, 3\ldots$

(We assume the mapping $F$ to be the identity transformation, and the knot vector to be "uniform", for its non multiple knots.)
B-splines of degree $p=1$ In case of piece-wise linear basis functions, we find ourselves in the first order Finite Element environment, where the Discrete Maximum Principle is known to hold (see Varga and others...).
Max Principle, case $p=2$

B-splines of degree $p=2$
When we use the parabolic B-spline basis and the uniform knot vector is considered, the matrix $A$ looks as follows:

$$A = \begin{pmatrix}
8 & -6 & -2 & 0 & 0 & 0 & \ldots & 0 \\
-6 & 8 & -1 & -1 & 0 & 0 & \ldots & 0 \\
-2 & -1 & 6 & -2 & -1 & 0 & \ldots & 0 \\
0 & -1 & -2 & 6 & -2 & -1 & \ldots & 0 \\
0 & \ldots & 0 & 0 & -1 & -1 & 8 & -6 \\
0 & \ldots & 0 & 0 & 0 & -2 & -6 & 8 \\
\end{pmatrix} \frac{h}{6},$$

where $h$ denotes the length $[t_{i+1} - t_i]$, for $t_{i+1} \neq t_i$. 

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matrix $K$ is:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-6 & 8 & -1 & -1 & 0 & 0 & \cdots & 0 \\
-2 & -1 & 6 & -2 & -1 & 0 & \cdots & 0 \\
0 & -1 & -2 & 6 & -2 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & -1 & -1 & 8 & -6 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
$$

Following Varga, we note that $K$ has negative off diagonal elements and is diagonal dominant, then it is an $M$–matrix iff there exists a vector $v$ with non-negative entries such that $Kv > 0$. 
Case $p=2$

Let $n \times n$ the dimension of $K$. Let $\mathbf{x} = \{x_i\}_{i=1}^n$ by $x_i = (i - 1)(n - i) + 1$.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-6 & 8 & -1 & -1 & 0 & 0 & \cdots & 0 \\
-2 & -1 & 6 & -2 & -1 & 0 & \cdots & 0 \\
0 & -1 & -2 & 6 & -2 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & -1 & -1 & 8 & -6 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{h}{6} \\
\frac{h}{6}
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
3n + 2 \\
n + 10 \\
12 \\
12 \\
\vdots \\
n + 10 \\
3n + 2 \\
1
\end{pmatrix}
\frac{h}{6} \cdot \mathbf{x} = 
\frac{h}{6}.

Hence $K \mathbf{x} > 0$,

The matrix $K$ is an $M$-matrix.
Its row sums are 1 for the first and the last rows and 0 for all the other rows, so $K \in \mathcal{N}_S$.
The matrix $K$ satisfies the Discrete Maximum Principle.
Case \( p = 3 \)

\( K_3 \) is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
-51 & 60 & 3/2 & -10 & -1/2 & 0 & 0 & \ldots & 0 \\
-19 & 3/2 & 27 & -4/3 & -47/6 & -1/3 & 0 & \ldots & 0 \\
-2 & -10 & -4/3 & 80/3 & -5 & -8 & -1/3 & \ldots & 0 \\
0 & -1/2 & -47/6 & -5 & 80/3 & -5 & -8 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & -1/2 & -10 & 3/2 & 60 & -51 \\
0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\hline
\end{pmatrix}
\]

\( h \frac{40}{40} \)

OUPS ! This is not an M-matrix !
2D case, $p=2$

Consider a quadratic total mesh $[0, 0, 0, 1/2, 1, 1, 1]^2$. We have only 4 internal degrees of freedom. The stiffness matrix in this case $A$ is a $(4 \times 4)$ matrix:

$$A = \frac{1}{45} \begin{pmatrix} 40 & 12 & 12 & 0 \\ 12 & 40 & 0 & 12 \\ 12 & 0 & 40 & 12 \\ 0 & 12 & 12 & 40 \end{pmatrix}.$$  \hfill \text{(15)}$$

Its inverse is:

$$A = \frac{1}{256} \begin{pmatrix} 369 & -135 & -135 & 81 \\ -135 & 369 & 81 & -135 \\ -135 & 81 & 369 & -135 \\ 81 & -135 & -135 & 369 \end{pmatrix},$$  \hfill \text{(16)}$$

which means that it is not a M-matrix (every M-matrix has a non-negative inverse matrix), and by Varga we do not have a maximum principle!

No Maximum Principle in 1D for $p \geq 3$ and in 2D and 3D for $p \geq 2$

Still IGA Domain Decomposition on non matching grids does work!
Green Function, 2D quadratic splines
Numerical Results

2D numerical results:

- Zooming DD convergence vs. analytical solutions
- Zooming DD convergence in a singular case
- Zooming: approximation of singular derivative
Analytical example, zoom and convergence

Let $\Omega$ be a circle of radius 3, $u = \sin(x^2 + y^2 - 9)$ is the solution of

$$-\Delta u = \sin(x^2 + y^2 - 9) - 4\cos(x^2 + y^2 - 9),$$

$$u|_{\partial\Omega} = 0.$$ 

Apply zooming by considering $\Omega$ as a union of an annulus and a square. $\Omega = \Omega_{\text{annulus}} \cup \Omega_{\text{square}}$.

Analytical example, zoom and convergence

The numerical solution:
Analytical example, zoom and convergence

L2-error for different degrees $p$ of the B-splines as a function of the mesh size.

![Graph showing L2-error for different degrees $p$. The graph includes lines for $p=2$, $p=3$, and $p=4$ with corresponding equations: $y = 42.397x^{3.2234}$, $y = 73.174x^{4.2014}$, and $y = 160.5x^{5.2852}$.](image)
Example of zoom: singularity at the corner

The problem:

\[ \triangle u = 0 \text{ in } \Omega, \]

\[ u = 0 \text{ on } \Gamma_1 \subset \Gamma_\Omega = \partial \Omega; \]

\[ u = \theta (\alpha - \theta) \text{ on } \Gamma_2 = \Gamma_\Omega \setminus \Gamma_1, \alpha = \frac{3 \pi}{2}, \]

\[ \Omega = \{(\rho, \theta) \in \mathbb{R} | 0 \leq \rho \leq 3; -\frac{\pi}{2} \leq \theta \leq \pi\}. \]
The exact solution is given by the series:

\[ u_{\text{ex}}(\rho, \theta) = \frac{9}{\pi} \sum_{n=1,3,5,...} \frac{1}{n^3} \left( \frac{\rho}{r} \right)^{\frac{2n}{3}} \sin \left( \frac{2n\theta}{3} \right) . \]
Example of zoom: singularity at the corner

Domain Decomposition: $\Omega = \Omega_{out} \cup \Omega_{zoom}$,
Example of zoom: singularity at the corner

Domain Decomposition: $\Omega = \Omega_{out} \cup \Omega_{zoom}$,
Example of zoom: singularity at the corner

Projection and non-homogeneous Dirichlet BC.
In this example we used the exact projection method and we imposed the Dirichlet boundary conditions by the least-squares method.

Exact solution is $r^{1/3}$, and our result coincides with this value.
The solution being in $H^{1+\epsilon}(\Omega)$ and not in $H^2(\Omega)$, we get a convergence rate that does not improve with $p$. 

![Graph showing L2_error (Givoli) with multiple lines and equations: $y = 0.0313x^{-1.326}$, $y = 0.0178x^{-1.369}$, $y = 0.0121x^{-1.4}$]
Numerical Results

3D examples:

- Heat and elasticity problems
- Parallelisation
3D examples
3D examples

Chain of cubes with the analytical solution $sin(x + y + z)$.
3D and Parallelisation

In order to solve "real 3D" examples we implemented a parallel version of the code using MatLab (not easy for shared data). Our solution:
use unrelated variables to perform the computations on each domain at every iteration and synchronize the solutions between the iterations.
180° hollow pipe, defined by 8 overlapping domains, elasticity model under uniform field force.

"Unit" Patch:
$180^\circ$ hollow pipe, defined by 8 overlapping domains, elasticity model under uniform field force.
180° hollow pipe, defined by 5 overlapping domains, elasticity model under uniform field force.
DD on unmatching grids/mapping provides a powerful and natural tool for IGA.

Parallelisation is easy and effective: on a quad core with 8 threads we have an acceleration factor of 4, using GEOPDEs.

To complete it we need to study:

- Dirichlet BC on trimmed surface patches.
- Preconditioning for very large problems.
- Incompressibility etc.
But Boolean operations are not limited to union and intersections, there also subtractions....

Hence we need to extend our method to Chimera type algorithm....
Thank you for your attention!