Properties of Minimal-Perimeter Polyominoes

Gil Ben-Shachar, Supervised by Prof. Gill Barequet

Department of Computer Science, Technion – Israel Institute of Technology

Introduction

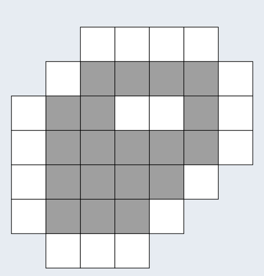
We study polyominoes with minimal perimeter for their area.
Main questions:

- What geometric and combinatorial properties minimal-perimeter polyominoes present?
- How many minimal-perimeter polyominoes exist for a given area n?

What is a Polyomino?

A polyomino is an edge-connected set of cells on the square lattice.

- The area of a polyomino is the number of cells of the polyomino.
- The perimeter of a polyomino is the set of cells adjacent to a polyomino cells. Denoted by $\mathcal{P}(Q)$.

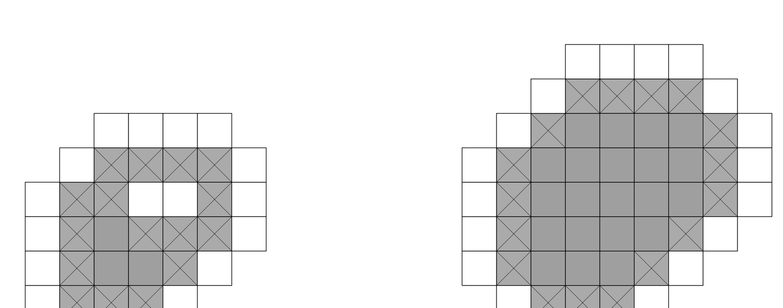


An example of a polyomino. The polyomino cells are gray, and perimeter cells are white.

Inflation of Polyominoes

The inflation of a polyomino is the union of the polyomino and its perimeter. Formally, the inflation of Q is

$$I(Q) = Q \cup \mathcal{P}(Q)$$



(a) Polyomino Q (b) $I(Q)$

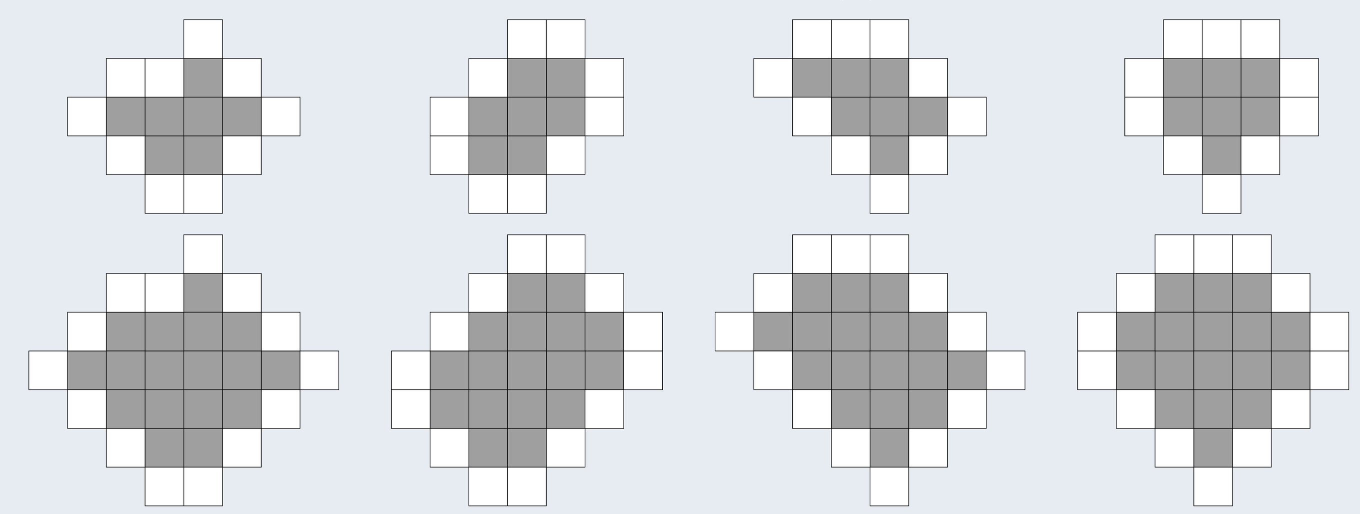
An example of a polyomino and the result of its inflation.

Inflation Induce a Bijection

Our main result is that inflation induce a bijection between sets of minimal-perimeter polyominoes.

- Q is a minimal-perimeter polyomino $\Rightarrow$ $I(Q)$ is a minimal-perimeter polyomino.
- If Q is a minimal-perimeter polyomino, and there exists a minimal-perimeter polyomino Q', such that $Q = I(Q')$, then $I(Q)$ is a minimal-perimeter polyomino as well.

Corollary: Let M_n be the set of minimal-perimeter polyominoes of size n, and $\epsilon(n)$ be the minimum perimeter of a polyomino with area n, then, $|M_n| = |M_{n+\epsilon(n)}|$



A demonstration of the conclusion. The first row contains all the elements of M_7, and the next row contains all the elements of M_{17}.

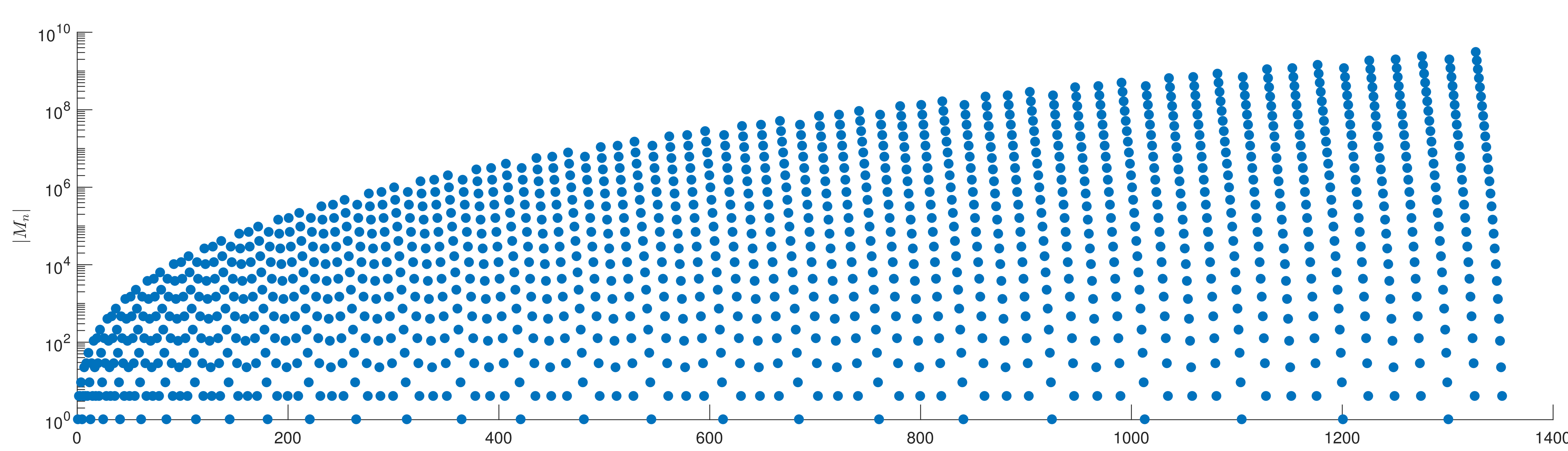
Chains of Minimal-Perimeter Polyominoes

We show that for any $k \in \mathbb{N}$, we have that $|M_n| = |M_{n+k\epsilon(n)+2k(k-1)}|$. We conclude, for example, that:

$$|M_{2477537}| = 4$$
$$|M_{2269511}| = 1464$$

Algorithmic Approach

Using the combinatorial properties of minimal-perimeter polyominoes we were able to develop a polynomial-time algorithm that counts the number of minimal-perimeter polyominoes of a given area. The algorithm computes the number of minimal-perimeter polyominoes of areas up to 1200 in a few hours. Beforehand only a few dozens of elements of this sequence were known.



Computed values of $|M_n|$.

Convergence of Inflation

Another result of this research is that given any polyomino, after a finite amount of inflation operations, the polyomino will become a minimal-perimeter polyomino. In other words, the process of inflating any polyomino repeatedly always converge to a minimal-perimeter polyomino.
We were also able to establish bounds on the convergence rate of this process.

Future Work

- Find a formula for $|M_n|$ for any n.
- Extend the ideas of this work to higher dimensions.
- Explore the implications of those results on related problems. Specifically, we are interested in the problem of counting the number of polyominoes with a given area n and some perimeter p. Solving this specific problem might shed some light on the problem of counting the number of polyominoes of a given area.

References

[1] Y. Altshuler, V. Yanovsky, D. Vainsencher, I.A. Wagner, and A.M. Bruckstein.
On minimal perimeter polyminoes.
In *Discrete Geometry for Computer Imagery*, pages 17–28. Springer, 2006.

[2] N. Sieben.
Polyominoes with minimum site-perimeter and full set achievement games.
European J. of Combinatorics, 29(1):108–117, 2008.

[3] A. Asinowski, G. Barequet, and Y. Zheng.
Enumerating polyominoes with fixed perimeter defect.
In *9th European Conf. on Combinatorics, Graph Theory, and Applications*, volume 61, pages 61–67, Vienna, Austria, August 2017. Elsevier.