Efficient Algorithms for Integer Programs with Two Variables per Constraint

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Technion

2VIP - 2 Variables per Constraint IP

Example:

$$\begin{aligned} \min \quad & 2x_1 + 5.6x_2 + 3.2x_3 \\ \text{s.t.} \quad & x_1 + x_2 \geq 3 \\ & -x_1 - x_3 \geq -4 \\ & x_2 - x_3 \geq 0 \\ & x_i \in \{0, \ldots, 4\} \end{aligned}$$

Definition:

$$\begin{array}{llll} \text{(2VIP)} \quad \min & \sum_{i=1}^{n} w_i x_i & w_i \geq 0 & \\ \text{s.t.} & a_k x_{i_k} + b_k x_{j_k} \geq c_k & a_k, b_k, c_k \in \mathbb{Z} & \forall k \in \{1, \ldots, m\} \\ & x_i \in \{\ell_i, \ldots, u_i\} & \ell_i, u_i \in \mathbb{N} & \forall i \in \{1, \ldots, n\} \\ & & U = \max_i \{u_i - \ell_i\} & \end{array}$$

Algorithms: Feasibility, 2-approximation.

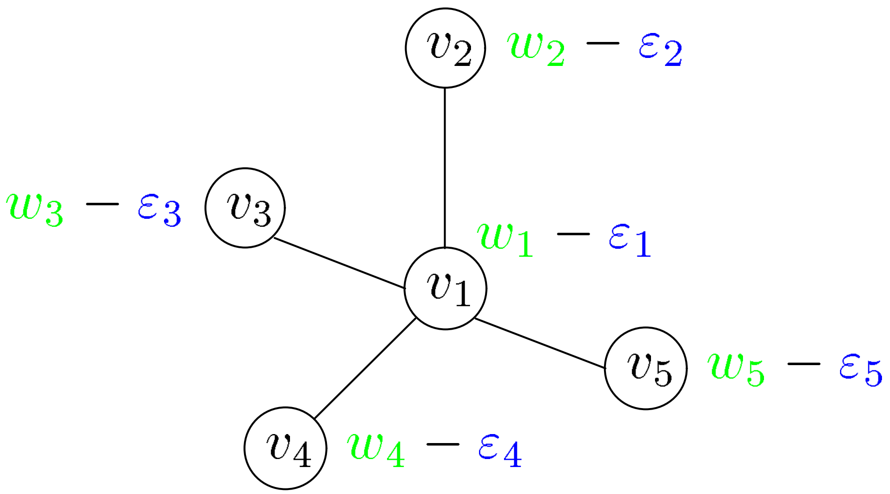
Minimum Weight Vertex Cover Problem

The Minimum WVC Problem for $G = (V, E)$ as a 2VIP system:

$$\begin{array}{lll} \min & \sum_{i=1}^{n} w_i x_i & \\ \text{s.t.} & x_i + x_j \geq 1 \ , & \forall (v_i, v_j) \in E \\ & x_i \in \{0, 1\} \ , & \forall v_i \in V \end{array}$$

$x_i = 1 \Leftrightarrow v_i \in \text{Cover}$

Local-Ratio intuition:



$\varepsilon = \varepsilon_1 = \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5$

Minimum Weight 2 Satisfiability

The min-2SAT Problem for a 2CNF formula φ as a 2VIP system:

$$
\begin{array}{llcl}
\min & \sum_{i=1}^{n} w_i x_i & & \\
\text{s.t.} & (x_i \vee x_j) \in \varphi & & x_i + x_j \geq 1 \\
& (x_i \vee \overline{x_j}) \in \varphi & \Longrightarrow & x_i - x_j \geq 0 \\
& (\overline{x_i} \vee \overline{x_j}) \in \varphi & & -x_i - x_j \geq -1 \\
& x_i \in \{\mathrm{T}, \mathrm{F}\} & & x_i \in \{0, 1\}
\end{array}
$$

Previous Results:

Feasibility	Linear time algorithm [EIS76]
2-approx	$O(nm)$ time and $O(n^2 + m)$ space algorithm [GP92]

2VIP - Results

Problem	Previous (time, space)	Our
2VIP Feas.	$O(m\,U), O(m\,U)$ [HMNT93] Using reduction to 2SAT*	$O(m\,U), O(m)$
2SAT Feas.	$O(m)$ [EIS76]	$O(m)$
2VIP 2-approx	$O(nm\,U^2 \log \frac{n^2\,U}{m}), O(m\,U)$ [HMNT93] Using reduction to 2SAT* and a Max-Flow algorithm	$O(nm\,U), O(m)$
2SAT 2-approx	$O(nm), O(n^2 + m)$ [GP92]	$O(nm), O(m)$

* The resulting 2CNF formula has $n\,U$ variables and $(m + n)\,U$ clauses.

Local-Ratio 2-approximation for Min-2SAT

Weight Reduction Phase:

$$x_1 = \mathrm{F} \quad \overset{x_1 \vee \overline{x_5}}{\Longrightarrow} \quad x_5 = \mathrm{F} \quad \overset{x_5 \vee x_2}{\Longrightarrow} \quad x_2 = \mathrm{T}$$

or

$$x_1 = \mathrm{T} \quad \overset{\overline{x_1} \vee \overline{x_4}}{\Longrightarrow} \quad x_4 = \mathrm{F} \quad \overset{x_3 \vee x_4}{\Longrightarrow} \quad x_3 = \mathrm{T}$$

Recursively solve for $(w_1 - \varepsilon_1, w_2 - \varepsilon_2, w_3 - \varepsilon_3, w_4, \ldots, w_n)$
where $\varepsilon = \varepsilon_1 + \varepsilon_3 = \varepsilon_2$

Solution Augmentation Phase:

Assign a **zero cost** (non-contradictory) partial assignment τ and recursively solve for the residual formula $\varphi(\tau)$

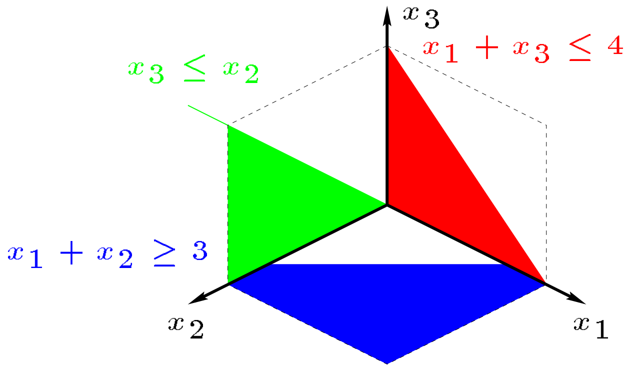
Constraint Propagation

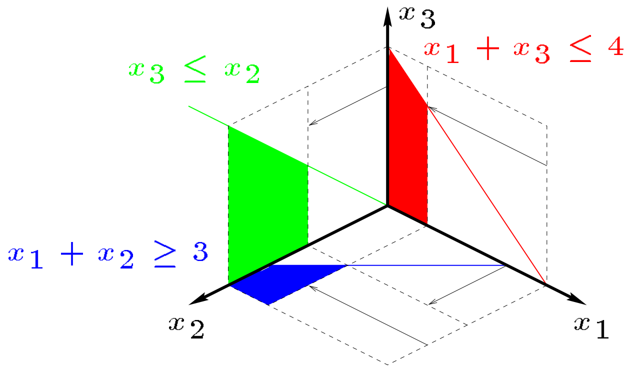
$$x_1 + x_2 \geq 3$$

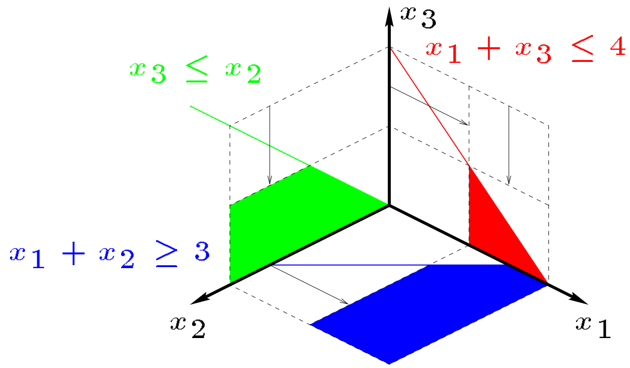
$$x_1 + x_3 \leq 4$$

$$x_3 \leq x_2$$

$$x_i \in \{0, \ldots, 4\}$$







$x_1 \leq 1 \overset{x_1+x_2 \geq 3}{\Longrightarrow} x_2 \geq 2$ $\qquad$ $x_1 \geq 2 \overset{x_1+x_3 \leq 4}{\Longrightarrow} x_3 \leq 2$

Q: Can we lose feasible solutions ? Yes.

Q: Can we lose all feasible solutions ? No.

From 2SAT to 2VIP

Feasibility:

Similar to 2SAT [EIS76] using **constraint propagation**.

Check in parallel:

- Impact of $x_t \leq \alpha$
- Impact of $x_t \geq \alpha + 1$

If one of these bounds leads to a non-contradictory bound change, make the change and continue.
Otherwise, the system is unsatisfiable.

2-approximation:

By extending the 2VIP definition we can use local-ratio arguments:

$$\begin{array}{rcl} x_t \leq \alpha & \rightarrow \cdots \rightarrow & x_i \geq \beta \\ x_t \geq \alpha + 1 & \rightarrow \cdots \rightarrow & x_j \geq \gamma \end{array}$$