

ON THE COMPLEXITY OF TIMETABLE AND MULTICOMMODITY FLOW PROBLEMS*

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Abstract. A very primitive version of Gotlieb's timetable problem is shown to be NP-complete, and therefore all the common timetable problems are NP-complete. A polynomial time algorithm, in case all teachers are binary, is shown. The theorem that a meeting function always exists if all teachers and classes have no time constraints is proved. The multicommodity integral flow problem is shown to be NP-complete even if the number of commodities is two. This is true both in the directed and undirected cases.

1. The timetable problem is NP-complete. The *timetable problem* (TT), which we shall discuss here, is a mathematical model of the problem of scheduling the teaching program of a school. In fact, it is a rather naive model since it ignores several factors which definitely play a role in practice [1]. However, we shall show that even a further restriction of the problem still leads to an NP-complete problem [2], [3].

DEFINITION (TT). Given the following data:

1. a finite set H (of hours in the week);
2. a collection $\{T_1, T_2, \dots, T_n\}$, where $T_i \subseteq H$; (there are n teachers and T_i is the set of hours during which the i th teacher is available for teaching);
3. a collection $\{C_1, C_2, \dots, C_m\}$, where $C_j \subseteq H$; (there are m classes and C_j is the set of hours during which the j th class is available for studying);
4. an $n \times m$ matrix R of nonnegative integers; (R_{ij} is the number of hours which the i th teacher is required to teach the j th class).

The problem is to determine whether there exists a meeting function

$$f(i, j, h) : \{1, \dots, n\} \times \{1, \dots, m\} \times H \rightarrow \{0, 1\}$$

(where $f(i, j, h) = 1$ if and only if teacher i teaches class j during hour h) such that:

- (a) $f(i, j, h) = 1 \Rightarrow h \in T_i \cap C_j$;
- (b) $\sum_{h \in H} f(i, j, h) = R_{ij}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$;
- (c) $\sum_{i=1}^n f(i, j, h) \leq 1$ for all $1 \leq j \leq m$ and $h \in H$;
- (d) $\sum_{j=1}^m f(i, j, h) \leq 1$ for all $1 \leq i \leq n$ and $h \in H$.

(a) assures that a meet takes place only when both the teacher and the class are available. (b) assures that the number of meets during the week between teacher i and class j is the required number R_{ij} . (c) assures that no class has more than one

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teacher at a time, and (d) assures that no teacher is teaching two classes simultaneously.

A teacher i is called a k -teacher if $|T_i| = k$; he is called *tight* if

$$|T_i| = \sum_{j=1}^m R_{ij},$$

that is, he must teach whenever he is available.

DEFINITION (RTT). RTT (the restricted timetable problem) is a TT problem with the following restrictions:

1. $|H| = 3$;
2. $C_j = H$ for all $1 \leq j \leq m$ (the classes are always available);
3. each teacher is either a tight 2-teacher or a tight 3-teacher;
4. $R_{ij} = 0$ or 1 for every $1 \leq i \leq n$ and $1 \leq j \leq m$.

Clearly both the TT and the RTT problem are in the NP class. We want to show that RTT is NP-complete. In that case TT is trivially NP-complete too. We recall that 3-SAT (satisfiability of a conjunctive normal form with 3 literals per clause) is NP-complete where 3-SAT is defined as follows: Given the data

1. a set of *literals* $X = \{x_1, x_2, \dots, x_l, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_l\}$,
2. a family of *clauses* D_1, D_2, \dots, D_k such that for every $1 \leq j \leq k$, $|D_j| = 3$ and $D_j \subseteq X$,

the problem is to determine whether there exists an assignment of values "true" and "false" to the literals, such that

- (a) exactly one of x_i and \bar{x}_i is assigned "true" while the other is assigned "false",
- (b) in each clause D_j there is at least one literal assigned "true".

THEOREM 1. 3-SAT \propto RTT.

Proof. The proof is by displaying a polynomially bounded reduction of the 3-SAT to RTT. In our construction, certain classes play the role of occurrences of literals x_i or \bar{x}_i in the clauses; the order in which some 2-teachers teach these classes indicates the truth value of the literals. All other classes and teachers are used in order to guarantee that this assignment of truth values satisfies conditions (a) and (b) above, and that all occurrences of a literal are assigned the same truth value.

Let p_i be the number of times the variable x_i appears in the clauses, i.e.,

$$p_i = \sum_{j=1}^k |D_j \cap \{x_i, \bar{x}_i\}|.$$

For each x_i we construct a set of $5 \cdot p_i$ classes which will be denoted by $C_{ab}^{(i)}$, where $1 \leq a \leq p_i$ and $1 \leq b \leq 5$ (we omit the superscript i whenever all classes used in the construction refer to the same i). In order to simplify the exposition, we shall use a graphic representation of the classes and teachers (see Fig. 1 for the structure corresponding to a single i). In our graphic representation the vertices denote class-hour combinations, where the rows signify the hours and the columns signify the classes. The hours are h_1, h_2 and h_3 . Now a 2-teacher who is available during hours h_1 and h_2 , and is supposed to meet once with $C_{a_1 b_1}$ and once with $C_{a_2 b_2}$ will be represented as shown in Fig. 2. The two diagonals show the only two

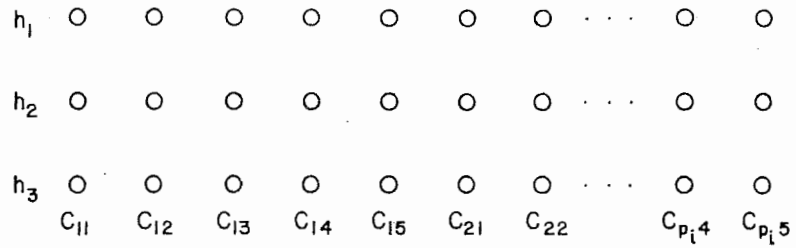


FIG. 1

ways possible to schedule this teacher. A 3-teacher who has to teach $C_{a_1b_1}$, $C_{a_2b_2}$ and $C_{a_3b_3}$ is denoted by a line with three arrows in the columns corresponding to these classes, as shown in Fig. 3. For every $1 \leq q \leq p_i$, we add two new classes, C'_{q1} and C''_{q1} with the structure shown in Fig. 4. There are three teachers described in the structure; two are 2-teachers and one 3-teacher. Since all these 3 teachers must teach during h_1 , the top 3 vertices, (h_1, C_{q1}) , (h_1, C'_{q1}) and (h_1, C''_{q1}) must be utilized.

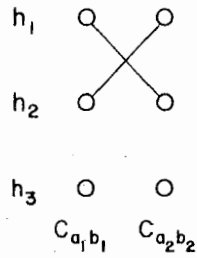


FIG. 2

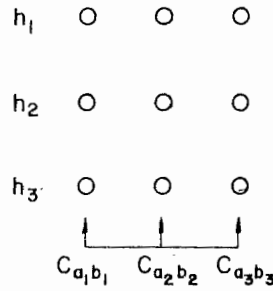


FIG. 3

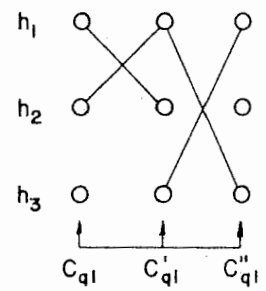


FIG. 4

However, we have a choice of utilizing exactly one of the vertices (h_2, C_{q1}) and (h_3, C_{q1}) , while leaving the other available; there are several ways to do this, as the reader may verify by himself. As far as the rest of our structure is concerned, the effect of this substructure is as follows: (h_1, C_{q1}) is taken and one of (h_2, C_{q1}) and (h_3, C_{q1}) is taken. Thus we shall delete (h_1, C_{q1}) from our diagrams.

Consider now the structure of teachers described in Fig. 5; it is intended to consistently assign truth values to all occurrences of x_i and \bar{x}_i in the clauses.

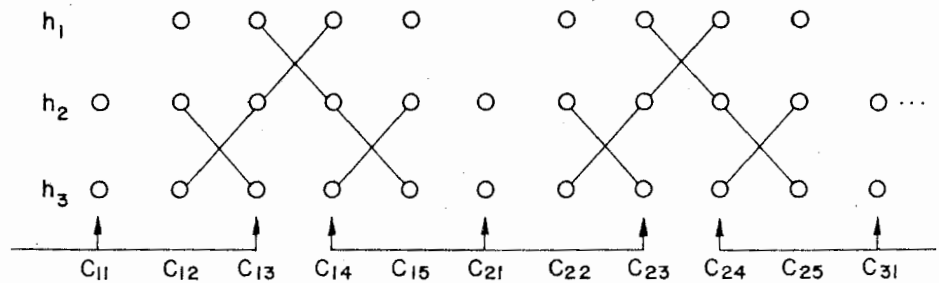


FIG. 5

Clearly, there is a 3-teacher assigned to classes $C_{p_i,4}$, $C_{1,1}$ and $C_{1,3}$; thus the structure described in Fig. 5 is circular. Consider now the p_i 2-teachers who are available during h_1 and h_2 , where the q th such teacher is assigned to classes $C_{q,3}$ and $C_{q,4}$. We claim that all these teachers must be scheduled in the same manner; that is, either all of them teach the $C_{q,3}$ classes during h_1 and the $C_{q,4}$ classes during h_2 , or all of them teach the $C_{q,3}$ classes during h_2 and $C_{q,4}$ classes during h_1 . Assume we have a schedule which does not satisfy this consistency condition. Then there must be a q such that the q th teacher teaches $C_{q,3}$ during h_2 and $C_{q,4}$ during h_1 , while the $(q + 1)$ st teacher¹ teaches the $C_{(q+1),3}$ during h_1 and $C_{(q+1),4}$ during h_2 . In this case, the 3-teacher who must teach $C_{q,4}$, $C_{(q+1),1}$ and $C_{(q+1),3}$ cannot be scheduled during h_1 —a contradiction.

We thus obtain, independently for each i , a uniform scheduling of all the 2-teachers who are available during h_1 and h_2 . The order in which these teachers teach $C_{q,3}$ and $C_{q,4}$ in the i th structure will be interpreted as the truth value of the variable x_i in the original 3-SAT problem.

We now add a few more 3-teachers, connecting the various i -structures, in order to guarantee that in each clause D_j , at least one literal gets the value "true". For every clause $D_j = \{\xi_1, \xi_2, \xi_3\}$, we assign a 3-teacher in the following way. He is assigned to one class for each of the three literals. If $\xi_1 = x_i$ and this is the q th appearance of this variable, then the corresponding class is $C_{q,2}^{(i)}$, while if $\xi_1 = \bar{x}_i$, the corresponding class is $C_{q,5}^{(i)}$. The classes corresponding to ξ_2 and ξ_3 are defined analogously.

This completes the definition of the RTT problem. The total number of classes defined is $21 \cdot k$, and the total number of teachers is $22 \cdot k$ ($15 \cdot k$ 2-teachers and $7 \cdot k$ 3-teachers). We claim that the given 3-SAT problem has a positive answer if and only if the RTT problem constructed above has a positive answer.

First, assume the 3-SAT problem has a positive answer. We use, now, the values of the literals in such an assignment to display a schedule for the constructed RTT problem—to prove that its answer is positive, too.

If x_i is assigned "true", then for every $1 \leq q \leq p_i$, the q th 2-teacher is scheduled to teach $C_{q,3}^{(i)}$ during h_1 and to teach $C_{q,4}^{(i)}$ during h_2 . Conversely, if x_i is assigned "false", then for every $1 \leq q \leq p_i$, the q th 2-teacher is scheduled to teach $C_{q,3}^{(i)}$ during h_2 and $C_{q,4}^{(i)}$ during h_1 .

In every clause D_j there is at least one literal assigned "true"; assume it is ξ . If $\xi = x_i$ and this is the q th appearance of this variable, then the 2-teacher who is supposed to teach $C_{q,2}$ and $C_{q,3}$ is scheduled to teach $C_{q,2}$ during h_3 and $C_{q,3}$ during h_2 .

(In our Fig. 6, the schedule assigned to each of the 2-teachers discussed so far is shown by a heavy solid line, and the choice we avoided is shown by a dashed line. A light solid line indicates that no choice has been made yet.) The 3-teacher of $C_{(q-1),4}$, $C_{q,1}$ and $C_{q,3}$ uses h_1 to teach $C_{(q-1),4}$, h_2 to teach $C_{q,1}$ and h_3 to teach $C_{q,3}$. (His meets are indicated by the circled vertices.) Finally, the 3-teacher corresponding to D_j uses h_2 to teach $C_{q,2}$. It remains to be shown that he can use h_1 and h_3 to teach the other two classes he is assigned to teach. Clearly, h_1 is never occupied

¹ Here $q + 1$ should be computed conventionally, except that $p_i + 1 = 1$, to fit the circular structure.

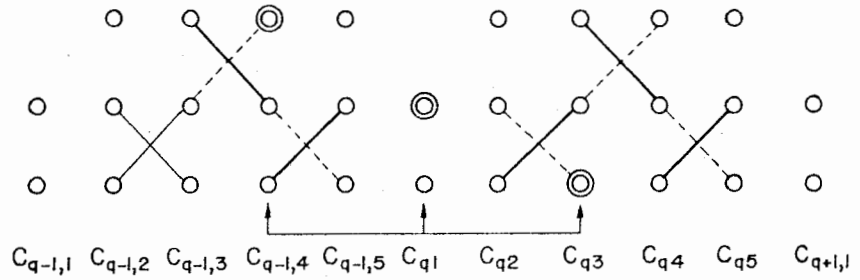


FIG. 6

by any other teacher in classes of types C_{a2} and C_{a5} . If $\xi' = x_r$ is another literal in D_j and it is "false", then the corresponding C_{a2} class must be taught during h_2 by the 2-teacher, and h_3 remains available. Also if $\xi' = \bar{x}_r$ and it is "false", then C_{a5} must be taught during h_2 by the 2-teacher and again h_3 remains available. Finally, if both remaining literals in D_j are "true", then for one of them, we do not follow the scheme used for ξ . For example, if $\xi' = x_r$, it is "true" and this is the a th appearance of this variable, then the 2-teacher teaches C_{a2} during h_2 and C_{a3} during h_3 . The 3-teacher teaches $C_{(a-1),4}$ during h_1 , C_{a1} during h_3 and C_{a3} during h_2 (as shown in Fig. 7). Thus h_3 remains available to teach C_{a2} , and the scheduling of the 3-teacher corresponding to D_j is now easy. The other cases are similar, and the reader may check them out for himself.

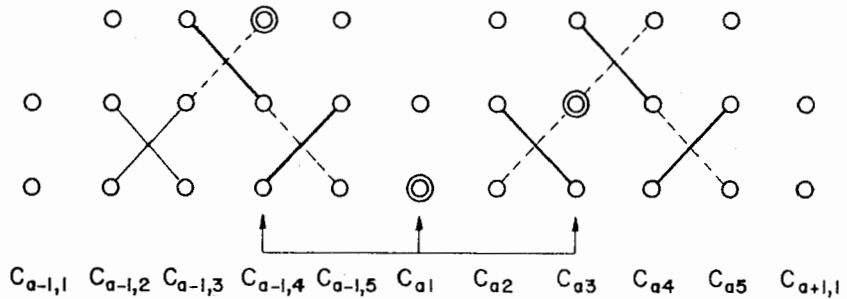


FIG. 7

Second, assume the answer to the constructed RTT problem is positive, and assume we have a legal scheduling. If in the structure of x_i the 2-teachers assigned to teach C_{q3} and C_{q4} teach C_{q3} during h_1 and C_{q4} during h_2 , then x_i is given the value "true", and if they teach C_{q3} during h_2 and C_{q4} during h_1 , then x_i is given the value "false". It remains to be shown that each clause $D_j = \{\xi_1, \xi_2, \xi_3\}$ contains at least one literal which is "true". If $\xi \in D_j$ and it is "false", then h_2 is used for teaching the corresponding class (a C_{a2} if $\xi = x_i$, and a C_{a5} if $\xi = \bar{x}_i$) by the 2-teacher that teaches it and the adjacent class (C_{a3} if $\xi = x_i$ and C_{a4} if $\xi = \bar{x}_i$). Thus, if all three literals are "false", the 3-teacher corresponding to D_j cannot have an assignment to teach its three classes, since he cannot use h_2 . Q.E.D.

2. The timetable problem with binary teachers is polynomially solvable. Consider the TT problem with the restriction that all teachers are 2-teachers. (A 1-teacher is of no interest.) We shall show that a simple branching procedure solves the problem in polynomial time, since the branching depth is limited.

Our algorithm will determine schedules for the teachers progressively. At a given stage, when part of the teachers have been scheduled we say that a teacher is *impossible* if he cannot be scheduled consistently; we say that he is *implied* if there is only one possible way to schedule him consistently with the schedules established so far.

ALGORITHM.

1. Set PHASE to 2.
2. If all teachers have been scheduled, halt with a positive answer.
3. If there is an unscheduled teacher who is impossible, go to step 7.
4. If there is no unscheduled implied teacher, go to step 6.
5. Let T_i be an unscheduled implied teacher. Temporarily schedule T_i as necessary and go to step 2.
6. Make all temporary schedules permanent. Let T_i be any unscheduled teacher. Arbitrarily choose a schedule for him and record this decision. Set PHASE to 1 and go to step 2.
7. If PHASE = 2, halt with a negative answer.
8. Reverse the schedule of the recorded teacher and undo all the temporary schedules. Set PHASE to 2 and go to step 3.

This algorithm clearly returns a positive answer only if a possible meeting function is constructed. It uses a limited backtracking since only one decision is ever recorded and possibly changed. It is less obvious that this limited backtracking is sufficient to discover a meeting function, if one exists.

Let a component of the evaluation be a set of teachers whose schedules gained permanency simultaneously (in step 6). The components may depend on arbitrary choices and on the order in which the teachers are considered. They are numbered consecutively according to their order of occurrence. For completeness, the set of teachers who are not scheduled or whose schedule had not been made permanent at the time the algorithm terminated is considered the last component.

LEMMA 1. *If T_i is a teacher of the last component, then none of the class-hours he may use is occupied by a teacher of a previous component.*

Proof. New components are started by entering step 6; but this occurs only when no teacher is implied. Since all teachers are binary, the lemma follows. Q.E.D.

The lemma implies that whenever the algorithm terminates with a negative answer, after trying both possible schedules for a certain teacher and all the schedules implied by it and failing, we can be sure that all the permanent schedules made before could not have hindered the situation, and thus the negative answer is conclusive.

It is worth noting here, that the technique of limited branching is applicable in other similar situations, such as the 2-SAT problem (i.e., the satisfiability problem for conjunctive normal forms with at most two literals per clause). Using appropriate data structures in order to find the implications of any decision made, and trying both decisions in step 6 in parallel (so that the quicker success stops the evaluation of the other possibility), it can be shown that the algorithm has

time complexity $O(n)$. Other known algorithms for the 2-SAT problem, such as the Davis and Putnam [4] algorithm (pointed out by Cook [2]) or an algorithm which follows from Quine's work [5] on the consensus (star) operation, have time complexity $O(n^2)$.

3. There is always a meeting function if all teachers and classes have no time constraints. The purpose of this section is to document a theorem which follows from the classical theory of matching in bipartite graphs [6].

We say that a given TT problem has no *time constraints* if for all $1 \leq i \leq n$ and $1 \leq j \leq m$ $T_i = C_j = H$; we say that it is *apparently feasible* if neither the teachers nor the classes are overloaded, i.e.:

$$(i) \text{ for all } 1 \leq i \leq n, \quad \sum_{j=1}^m R_{ij} \leq |H|,$$

$$(ii) \text{ for all } 1 \leq j \leq m, \quad \sum_{i=1}^n R_{ij} \leq |H|.$$

Clearly the condition that a TT problem be apparently feasible is necessary for the existence of a meeting function, but is not sufficient.

Our purpose is to prove the following theorem:

THEOREM 2. *If a TT problem is apparently feasible and has no time constraints, then it has a meeting function.*

Proof. First let us define the following quantities:

$$r = \sum_{i=1}^n \sum_{j=1}^m R_{ij}, \quad h = |H|,$$

$$v = m - \left\lfloor \frac{r}{h} \right\rfloor, \quad \mu = n - \left\lfloor \frac{r}{h} \right\rfloor.$$

Now, define a bipartite multi-graph $G(X, Y, E)$ in the following way:

$$X = \{x_1, x_2, \dots, x_n\} \cup \{\xi_1, \xi_2, \dots, \xi_v\},$$

$$Y = \{y_1, y_2, \dots, y_m\} \cup \{\eta_1, \eta_2, \dots, \eta_\mu\}.$$

E is a set of edges connecting between vertices of X and vertices of Y constructed as follows. For every $1 \leq i \leq n$ and $1 \leq j \leq m$, we put R_{ij} parallel edges between x_i and y_j . Next, for each $1 \leq i \leq n$, we complete the degree² of x_i to be exactly h by putting $h - \sum_{j=1}^m R_{ij}$ edges between x_i and vertices of $\{\eta_1, \eta_2, \dots, \eta_\mu\}$; it does not matter to which of these vertices these edges are connected provided the degree of each η_k never exceeds h . Also, for each $1 \leq j \leq m$, we complete the degree of y_j to be exactly h by putting $h - \sum_{i=1}^n R_{ij}$ edges between y_j and vertices of $\{\xi_1, \xi_2, \dots, \xi_v\}$, again taking care that the degree of each ξ_i never exceeds h . Finally, we complete the degree of the vertices in $\{\xi_1, \xi_2, \dots, \xi_v\}$ and $\{\eta_1, \eta_2, \dots, \eta_\mu\}$ to be exactly h , too, by putting edges from any ξ_i to any η_k which both have a lower degree.

It remains to show that this definition is proper in the sense that all the conditions it implies are easily met.

² The degree of a vertex is the number of edges incident to it.

The number of edges we construct in the completion of the degrees of x_1, x_2, \dots, x_n is $n \cdot h - r$. Thus we can do this if $\mu \cdot h \geq n \cdot h - r$, and μ satisfies this inequality. Similarly, ν satisfies the condition for the possibility of the completion of the degrees of y_1, y_2, \dots, y_m . Finally, the number of edges required to complete the degrees of $\xi_1, \xi_2, \dots, \xi_\nu$ is $\nu \cdot h - (m \cdot h - r)$, which is equal to $r - \lfloor r/h \rfloor \cdot h$. (This is the remainder of r upon division by h .) Similarly, the number of edges required to complete the degrees of $\{\eta_1, \eta_2, \dots, \eta_\mu\}$ is the same. Thus, the last part of the construction raises no difficulties, either.

Next, let $\Gamma(A)$, where $A \subseteq X$, be the set of vertices $B \subseteq Y$ such that there is an edge $a \rightarrow b \in E$, where $a \in A$ and $b \in B$.

LEMMA 2. For every $A \subseteq X$, $|\Gamma(A)| \geq |A|$.

Proof. There are $h \cdot |\Gamma(A)|$ edges incident to $\Gamma(A)$ in G . This includes all the edges which are incident to A . Thus

$$h \cdot |\Gamma(A)| \geq h \cdot |A|. \quad \text{Q.E.D.}$$

Lemma 2 assures that Hall's condition holds, and thus, by Hall's theorem [6], there is a set of $n + \nu (= m + \mu)$ edges, no two of which have a common endpoint. We now use this set of edges M (which is commonly called a complete match of X to Y) to define the meeting function for the first hour $h_1 \in H$; if $x_i \rightarrow y_j \in M$, then $f(i, j, h_1) = 1$; otherwise $f(i, j, h_1) = 0$. Clearly conditions (c) and (d) hold for h_1 . Next we remove M from E . The new graph has degree $h - 1$ for all its vertices, and as in Lemma 2, Hall's condition holds again. This assures the existence of another complete match M' of X to Y , and we can use it to define $f(i, j, h_2)$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. We repeat this until, by the h th application, all E 's edges have been used. This assures that condition (b) holds. Thus the proof of Theorem 2 is complete. Q.E.D.

The technique used here is an easy generalization of the one classically used to prove the school dance theorem. (See, for example, [7, Example 2, p. 92].) Since the proof is constructive and a complete match of X to Y can be obtained in polynomial time (Hopcroft and Karp [8]), this technique can be used in order to find an appropriate scheduling in polynomial time rather than just proving its existence.

4. The two-commodity integral flow problem is NP-complete. Knuth (see [9]) has shown that the multicommodity integral flow problem is NP-complete. His reduction, from the satisfiability problem, uses as many commodities as there are clauses.

We present a reduction of the satisfiability problem to the two-commodity integral flow in directed graphs (D2CIF), and in turn, a reduction of the D2CIF problem to the U2CIF (the undirected version).

DEFINITION (D2CIF). Given the following data:

1. $G(V, E)$ a directed finite graph; a directed edge from u to v is denoted $u \rightarrow v$;
2. a capacity function $c: E \rightarrow N$, where N is the set of nonnegative integers;
3. vertices s_1 and s_2 (not necessarily distinct) which are called the *sources*;
4. vertices t_1 and t_2 (not necessarily distinct) which are called the *terminals*;
5. two nonnegative integers R_1 and R_2 which are called the *requirements*.

The problem is to determine whether there exist two flow functions f_1 and f_2 , both $E \rightarrow N$, such that

- (a) for every $u \rightarrow v \in E$, $f_1(u \rightarrow v) \geq 0$, $f_2(u \rightarrow v) \geq 0$ and $f_1(u \rightarrow v) + f_2(u \rightarrow v) \leq c(u \rightarrow v)$;

intuitively, the commodities flow along the directed edge $u \rightarrow v$ from u to v . The total flow along an edge is bounded from above by the capacity of the edge;

- (b) for each commodity $i \in \{1, 2\}$ and each vertex $v \in V - \{s_i, t_i\}$

$$\sum_{u \rightarrow v \in E} f_i(u \rightarrow v) = \sum_{v \rightarrow w \in E} f_i(v \rightarrow w);$$

this is the conservation rule which states that for each commodity the amount of flow which enters a vertex equals the flow which emanates from it;

- (c) for each commodity $i \in \{1, 2\}$, let the total flow be

$$F_i = \sum_{s_i \rightarrow v \in E} f_i(s_i \rightarrow v) - \sum_{v \rightarrow s_i \in E} f_i(v \rightarrow s_i);$$

then it is required that

$$F_i \geq R_i.$$

A flow problem is *simple* if the capacities of all edges are equal to one.

THEOREM 3. *Simple D2CIF is NP-complete.*

Proof. It suffices to show: satisfiability \propto simple D2CIF.

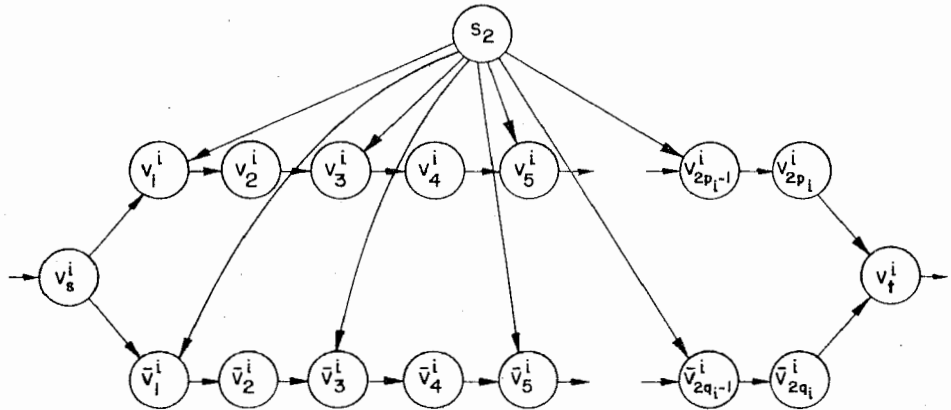


FIG. 8

Let the clauses of the satisfiability problem be D_1, \dots, D_k and $x_1, \dots, x_l, \bar{x}_1, \dots, \bar{x}_l$ be the literals. For each variable x_i , we construct a lobe as shown in Fig. 8. (Here p_i is the number of occurrences of x_i in the clauses and q_i is the number of occurrences of \bar{x}_i .) The capacity of all edges is 1. The lobes are connected to one another in series: v_i^i is connected to v_s^{i+1} , s_1 is connected to v_s^1 and v_t^i to t_1 . s_2 is connected to all the vertices v_j^i and \bar{v}_j^i where j is odd. In addition, there are

vertices D_1, D_2, \dots, D_k and an edge from each to t_2 . For the j th occurrence of x_i (\bar{x}_i), there is an edge from v_{2j}^i (\bar{v}_{2j}^i) to the D_j , in which it occurs. The requirements are $R_1 = 1$ and $R_2 = k$.

(a) Assume that there exist flow functions f_1 and f_2 which satisfy the requirements. Clearly, $F_1 = 1$ and $F_2 = k$. The unit of the first commodity flow must pass through all lobes. Define x_i to be "true" if and only if the first commodity flow passes through the lower path of the i th lobe. In this case, flow of the second commodity may pass through the upper part of the lobe to all the clauses which contain x_i . Since $F_2 = k$, through each vertex D_j there is a unit flow of the second commodity. Assume that this unit of flow comes from the i th lobe. If it comes from the upper part of the lobe, then $x_i \in D_j$ and the first commodity must flow through the lower part of the lobe. Thus x_i is "true" and D_j is satisfied.

If the flow comes from the lower part of the lobe, a similar argument holds. This completes the proof that the expression is satisfiable.

(b) If the expression is satisfiable, we send the first commodity flow through the lower path of the i th lobe if and only if x_i is "true". Since each clause D_j contains at least one literal x_i or \bar{x}_i which is "true", the second commodity passes through the upper or lower path depending on whether x_i or \bar{x}_i is "true".

Thus both requirements are met. Q.E.D.

Next, we show that U2CIF [10] is NP-complete, too. The definition of U2CIF is similar to that of D2CIF except that the graph is undirected. Denoting an undirected edge between u and v as $u - v$, its capacity is $c(u - v)$. However, the flow has a direction. If the flow is from u to v $f_i(u - v)$ is positive and $f_i(v - u)$ is its negation. (Note that $c(u - v) = c(v - u) \geq 0$.) Condition (a) changes into

$$|f_1(u - v)| + |f_2(u - v)| \leq c(u - v), \quad u - v \in E,$$

implying that the total flow in both directions is less than the capacity.

As before, condition (b) assures that for each $v \in V - \{s_i, t_i\}$ the total flow of commodity i entering v is equal to the total flow of commodity i emanating from v , i.e.,

$$\sum_{u-v \in E} f_i(u - v) = 0.$$

Let the total i th commodity flow be $F_i = \sum_{s_i - u \in E} f_i(s_i - u)$. Condition (c) states that $F_i \geq R_i$.

THEOREM 4. *Simple U2CIF is NP-complete.*

Proof. It suffices to show:

$$\text{simple D2CIF} \propto \text{simple U2CIF}.$$

First we change the directed graph $G(V, E)$ as follows: we add four new vertices $\bar{s}_1, \bar{s}_2, \bar{t}_1$ and \bar{t}_2 to serve as the two new sources and terminals, respectively. We connect \bar{s}_1 to s_1 via R_1 parallel edges and t_1 to \bar{t}_1 via R_1 parallel edges. Similarly, \bar{s}_2 is connected to s_2 and t_2 to \bar{t}_2 via R_2 parallel edges in each case. Vertices s_1, s_2, t_1 and t_2 are now subject to the conservation rule and the requirements are the same. Clearly, the requirements can be met in the new graph $G'(E', V')$ if and only if they can be met in the original one. Also, without loss of generality, we may assume that $R_1 + R_2 \leq |E|$, or obviously the requirements cannot be met.

Thus these changes can only expand the data describing the problem linearly. Now we proceed to construct the undirected network from the new directed network.

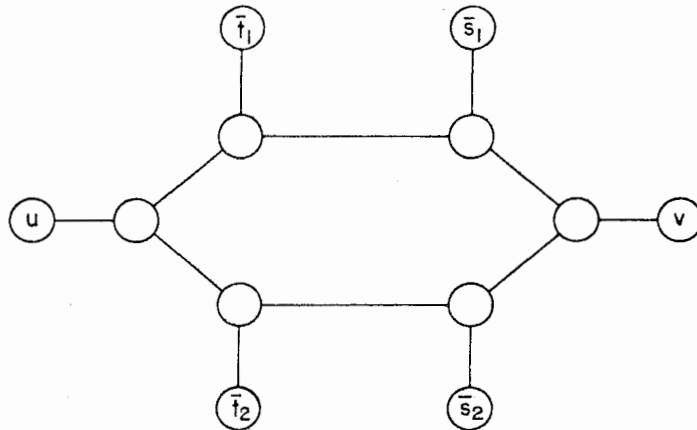


FIG. 9

Each edge $u \rightarrow v$ of G' is replaced by the construct shown in Fig. 9. (u or v may be one of the sources or terminals.) Only the unlabeled vertices of the construct are new and do not appear elsewhere in the graph.

It remains to be shown that the requirements can be met in the directed network if and only if the requirements $R'_1 = R_1 + |E'|$ and $R'_2 = R_2 + |E'|$ can be met in the undirected network.

First assume that the requirements of the directed network can be met. Initially, flow one unit of each commodity through each one of the edge-constructs, as shown in Fig. 10. This yields $F_1 = |E'|$ and $F_2 = |E'|$. Next, if $u \rightarrow v$ is used in G' to flow one unit of the first commodity, then we change the flows in the edge-construct as shown in Fig. 11. The case of the second commodity flowing through

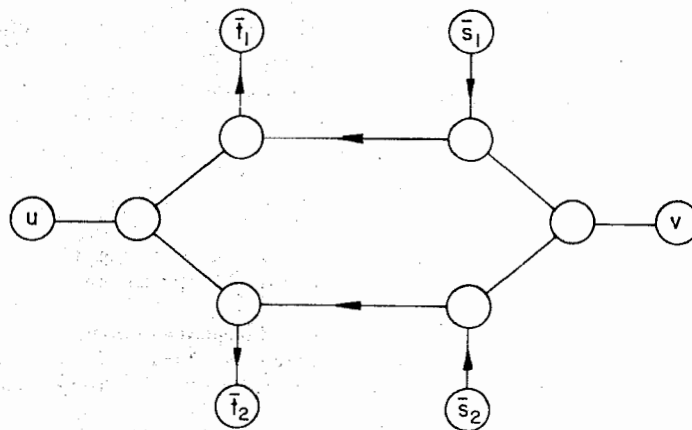


FIG. 10

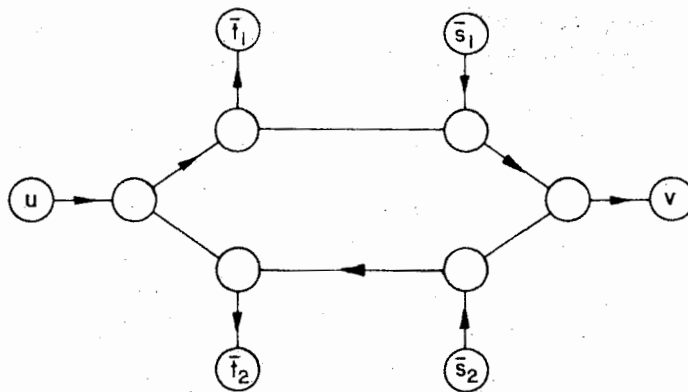


FIG. 11

$u \rightarrow v$ in G' is handled similarly. It is easy to see that R'_1 and R'_2 are now met in the undirected graph.

Now assume we have a flow in the undirected graph satisfying the requirements R'_1 and R'_2 . Since the number of edges incident to \bar{s}_i (\bar{t}_i) is R'_i , all these edges are used to emanate (inject) i th commodity flow from (into) \bar{s}_i (\bar{t}_i). The flow through each edge-construct must therefore be in one of the following patterns:

1. as in Fig. 10;
2. as in Fig. 11;
3. as in Fig. 11, for the second commodity.

We can now use the following flow through $u \rightarrow v$ in G' : If the $u \rightarrow v$ construct is of pattern 1, then $f_1(u \rightarrow v) = f_2(u \rightarrow v) = 0$. If it is of pattern 2, then $f_1(u \rightarrow v) = 1$ and $f_2(u \rightarrow v) = 0$, etc. Clearly this defines a legal flow for G' which meets the requirements. Q.E.D.

It is easy to see that the multicommodity integral flow problems, as we have defined them, are easily reducible to the version in which we have only one (total) requirement, i.e., $F_1 + F_2 \geq R$. Thus, the latter versions are NP-complete, too. Also, the completeness of the above problems imply the completeness of the two-commodity integral flow problems with arbitrary capacities for both the directed and the undirected case. Also, the completeness of $m \geq 2$ -commodity integral flow problems follows.

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