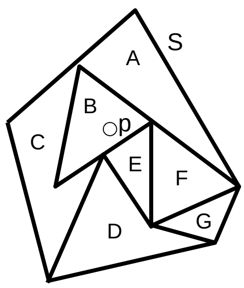
Computational Geometry

Chapter 6

Point Location

Problem Definition

- Preprocess a planar map *S.* Given a query point *p*, report the face of *S* containing *p.*

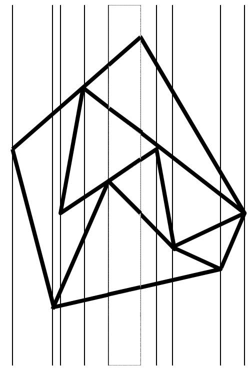


- **Goal:** O(n)-size data structure that enables O(log n) query time.
- **Application:** Which state is Baltimore located in?
  Answer: Maryland
- **Trivial Solution:** O(n) query time, where n is the complexity of the map. (Question: Why is the query time only O(n)?)

Naïve Solution

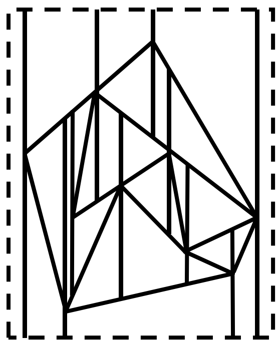
- Draw vertical lines through all the vertices of the subdivision.
- Store the *x*-coordinates of the vertices in an ordered binary tree.
- Within each slab, sort the segments separately along *y*.
- Query time: O(log *n*).
- **Problem**: Too delicate subdivision, of size $\Theta(n^2)$ in the worst case.

  (Give such an example!)

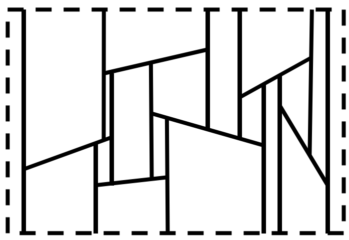


The Trapezoidal Map

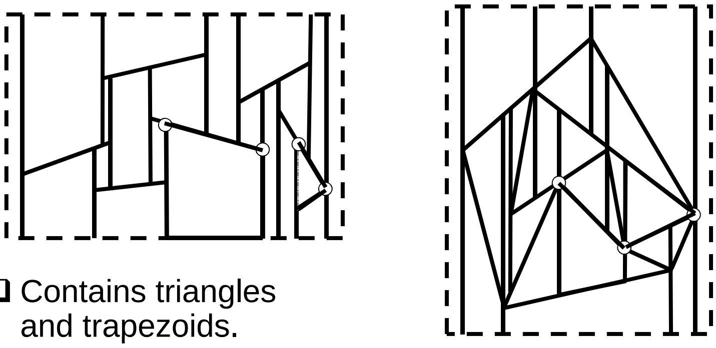
- Construct a bounding box.
- Assume general position: unique x coordinates.
- Extend upward and downward the vertical line from each vertex until it touches another segment.



- This works also for noncrossing line segments.



Properties



- Contains triangles and trapezoids.
- Each trapezoid or triangle is determined:
  - By two vertices that define vertical sides; and
  - By two segments that define nonvertical sides.
- A *refinement* of the original subdivision.

Notation

Every trapezoid (or triangle) Δ is defined by

- Left(Δ): a segment endpoint (right or left);
- Right(Δ): a segment endpoint (right or left);
- Top(Δ): a segment;
- Bottom(Δ): a segment.

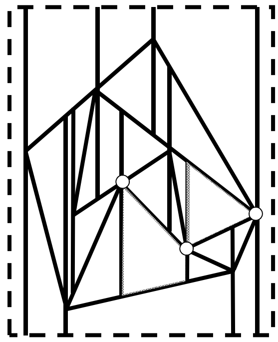
Complexity

- ❑ **Theorem** (linear complexity)**:**
  A trapezoidal map of n segments contains at most $6n+4$ vertices and at most $3n+1$ faces.
- ❑ **Proof:**
  1. Vertices:

$$\underset{\text{original}}{\underset{\uparrow}{2n}} + \underset{\text{extensions}}{\underset{\uparrow}{4n}} + \underset{\text{box}}{\underset{\uparrow}{4}} = 6n + 4$$

  2. Faces: Count Left(Δ).

$$\underset{\text{left e.p.}}{\underset{\uparrow}{2n}} + \underset{\text{right e.p.}}{\underset{\uparrow}{n}} + \underset{\text{box}}{\underset{\uparrow}{1}} = 3n + 1$$



Question:

Why does the proof hold for “degenerate” situations?

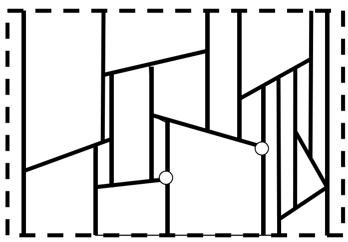
Map Data Structure

- ❑ Possibly by DCEL.

An alternative:

For each trapezoid store:

- ❑ The vertices that define its right and left sides;
- ❑ The top and bottom segments;
- ❑ The (up to *two*) neighboring trapezoids on right and left;
- ❑ (Optional) The neighboring trapezoids from above and below. This number might be linear in n, so only the leftmost of these trapezoids is stored.

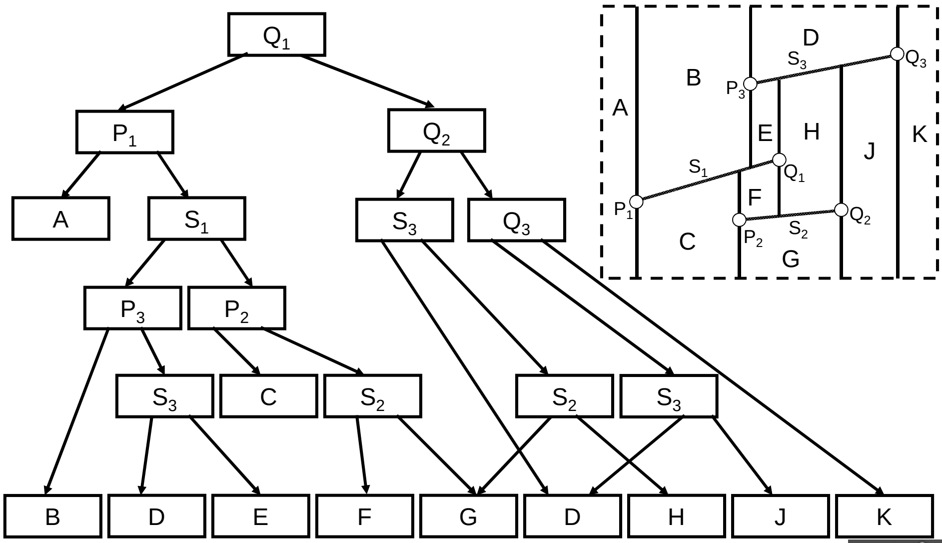


Note: Computing any trapezoid from the trapezoidal structure can be done in constant time.

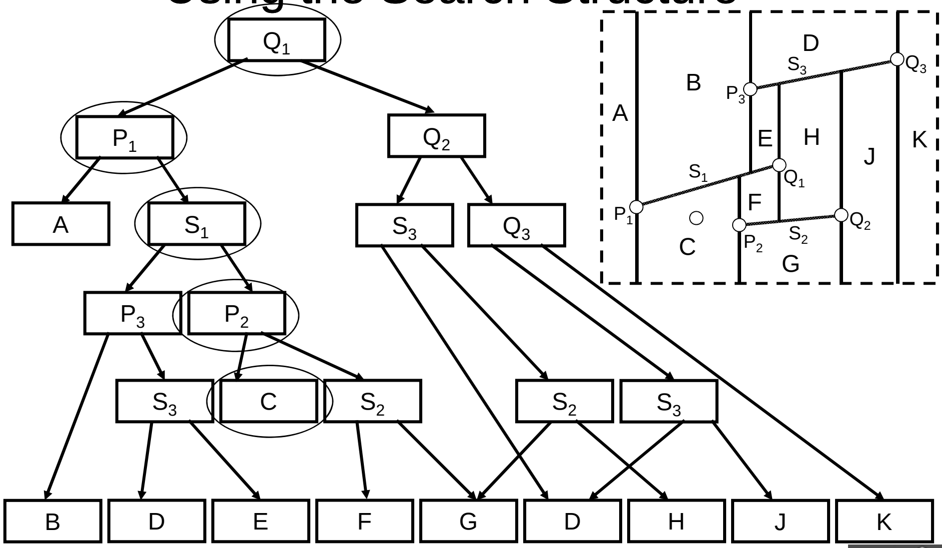
Search Structure: Branching Rules

- Query point *q*, search-structure node *s*.
- *s* is a segment **endpoint**:
  - *q* is to the **right** of *s*: go **right**;
  - *q* is to the **left** of *s*: go **left**;
- *s* is a segment:
  - *q* is **below** *s*: go **right**;
  - *q* is **above** *s*: go **left**;

The DAG Search Structure

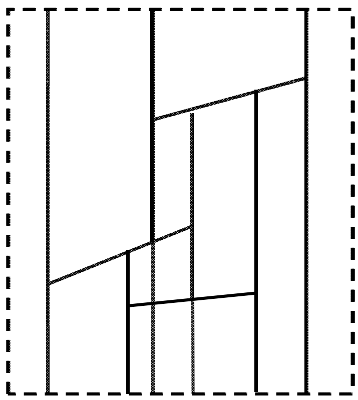


Using the Search Structure



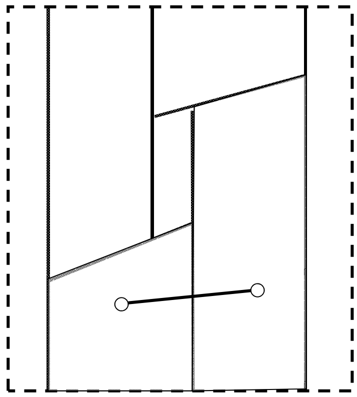
Search Structure: Construction

- ❑ Find a Bounding Box.
- ❑ Randomly permute the segments.
- ❑ Insert the segments one by one into the map.
- ❑ Update the map and search structure in each insertion.
- ❑ The size of the map is $\Theta(n)$. (This was proven earlier.)
- ❑ The size of the search structure depends on the order of insertion.



Updating the Structures (High Level)

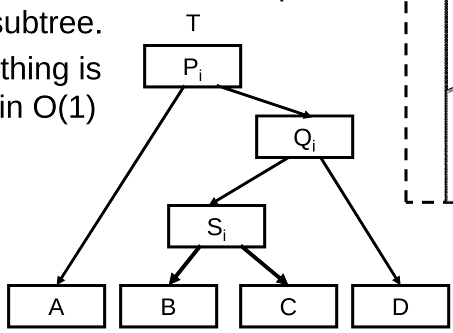
- Find **in the existing structure** the face that contains the left endpoint of the new segment. (*)
- Find all other trapezoids intersected by this segment by moving to the right. (In each move choose between two options: Up or Down.)
- Update the map M_i and the search structure D_i.

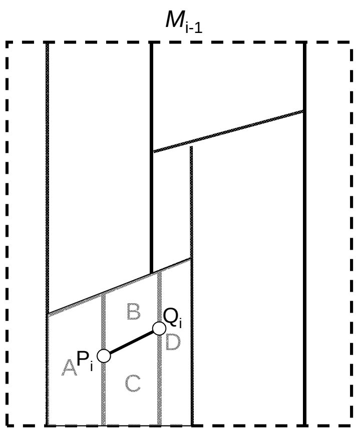


(*) Note: Since endpoints may be shared by segments, we need to consider its segment while searching.

Update: Simple Case

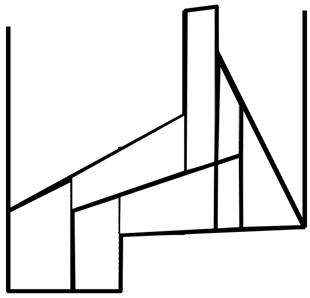
- The segment is contained entirely in one trapezoid.
- In M_{i-1}: Split the trapezoid into four trapezoids.
- In D_{i-1}: The leaf will be replaced by a subtree.
- Everything is done in O(1) time.





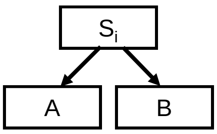
Update *M*: General Case

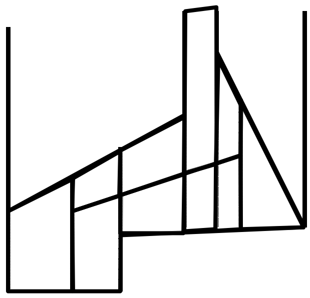
- General Case: The i^{th} segment intersects with $k_i>1$ trapezoids.
- Split trapezoids.
- Merge trapezoids that can be united.
- Total update time: $O(k_i)$.



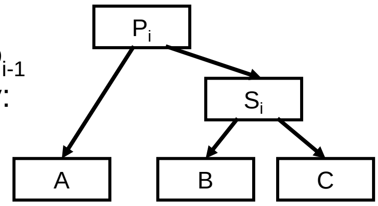
Updating *D*: Split

- Each *inner* trapezoid in D_{i-1} is replaced by:



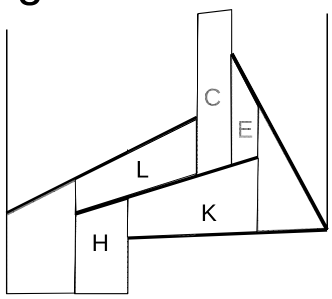


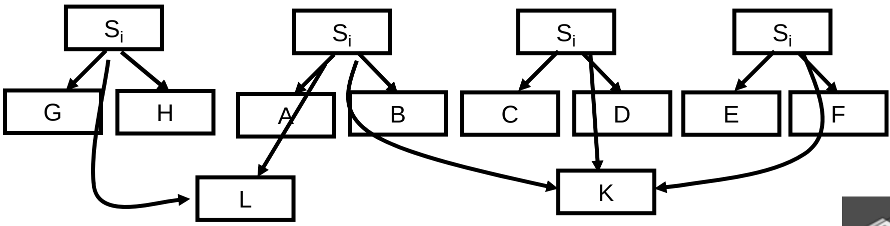
- Each *outer* (e.g., left) trapezoid in D_{i-1} is replaced by:



Updating D: Merge

- ❑ Leaves are eliminated and replaced by one common leaf.
- ❑ Total update time: $O(k_i)$.





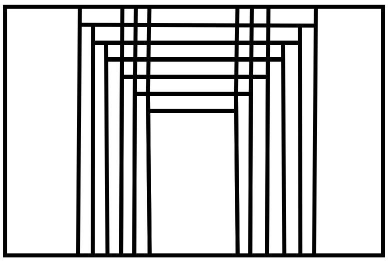
Construction: Worst-Case Analysis

- ❑ Each segment adds trees of depth at most (4-1=) 3, so the depth of D_i is at most 3i.
- ❑ Query time (depth of D_i): $O(i)$, $\Theta(i)$ in the worst case.
- ❑ The i^{th} segment, s_i, may intersect with **O(*i*)** trapezoids (**Θ(*i*)** in the worst case)!
- ❑ The size of D and its construction time are then bounded from above by

$$\sum_{i=1}^{n} O(i) = O(n^2) \ .$$

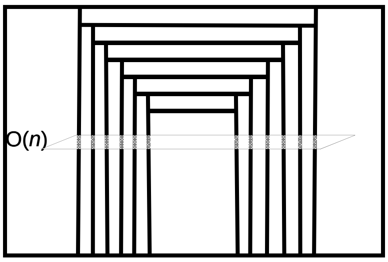
Construction: Worst-Case Analysis (cont.)

Worst-case example:



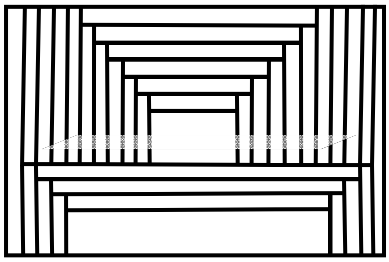
Construction: Worst-Case Analysis (cont.)

Worst-case example:



Construction: Worst-Case Analysis (cont.)

Worst-case example:



The size of D and its construction time is in the worst case.

$$\sum_{i=1}^{\frac{n}{2}} \Theta(1) + \sum_{i=\frac{n}{2}+1}^{n} \Theta(n) = \Theta(n^2)$$

Average-Case Analysis

- We first look for the expected depth of D.
- q: A point, to be searched for in D.
- p_i: The probability that a new vertex was created in the path leading to q in the i^{th} iteration.

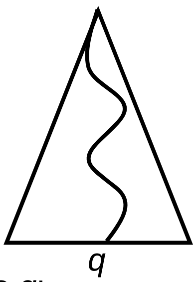
Compute p_i by backward analysis:

- $\Delta_q(M_{i-1})$: The trapezoid containing q in M_{i-1}.
- Since a new vertex was created, $\Delta_q(M_i) \neq \Delta_q(M_{i-1})$.
- Delete s_i from M_i.

p_i = Prob$[\Delta_q(M_i) \neq \Delta_q(M_{i-1})] \leq 4/i$. (Why?)

Expected Depth of *D*

- x_i: The number of vertices **created in the *i*th** iteration in the path leading to the leaf *q*.



- The expected length of the path leading to *q*:

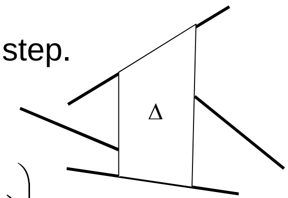
$$\mathrm{E}\left[\sum_{i=1}^{n} x_i\right] = \sum_{i=1}^{n} \mathrm{E}[x_i] \le \sum_{i=1}^{n} (3p_i) \le \sum_{i=1}^{n} \tfrac{12}{i} = \mathrm{O}(\log n).$$

Expected Size of *D*

- Define an indicator

$$\delta_i(\Delta, s) = \begin{cases} 1 & \Delta \text{ disappears from } M_i \text{ if } s \text{ is removed} \\ 0 & \text{otherwise} \end{cases}$$

- k_i: Number of leaves created in the *i*th step.
- S_i: The set of the first *i* segments.
- Average on *s*:



$$\mathrm{E}[k_i] = \tfrac{1}{i}\sum_{s\in S_i}\left(\sum_{\Delta\in M_i}\delta_i(\Delta, s)\right) = \tfrac{1}{i}\left(\sum_{s\in S_i}\sum_{\Delta\in M_i}\delta_i(\Delta, s)\right)$$

$$\le \tfrac{1}{i}(4\,|M_i|) \quad \text{(same backward analysis)}$$

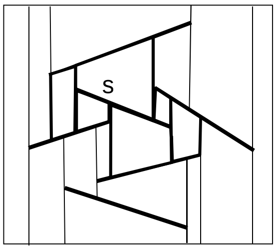
$$= \tfrac{\mathrm{O}(i)}{i} = \mathrm{O}(1).$$

Expected Size of *D* (cont.)

- k_i-1: Number of internal nodes created in the i^{th} step.
- Total size:

$$O(n) + E\left(\sum_{i=1}^{n}(k_i - 1)\right) = O(n) + E\left(\sum_{i=1}^{n} k_i\right) = O(n).$$

↑ leaves ↑ internal



Expected Construction Time of *D*

$$\sum_{i=1}^{n}\left(O(E(\text{depth}_i)) + O(E[k_i])\right)$$

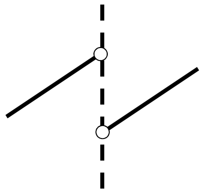
$$= \sum_{i=1}^{n}(O(\log i) + O(1)) = O(n \log n)$$

Finding the first trapezoid

The rest of the work in the i^{th} step

Handling Degeneracies

- What happens if two segment endpoints have the same *x* coordinate?
- Use a shearing transformation: $\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+\varepsilon y \\ y \end{pmatrix}$



- Higher points will move more to the right.
- ε should be small enough so that this transform will not change the order of two points with different *x* coordinates.
- In fact, there is no need to shear the plane. Comparison rules **mimic** the shearing.
- Prove: The entire algorithm remains correct.